ON ELECTRODYNAMICS I, II AND III

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A translation of: **Zur Electrodynamik.**
I: Zwei Formen der kleinsten Action in der Elektronentheorie;
II: Die elementarische elektodynamische Kraft;
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CONTENTS

Part 1. Two variations of the principle of least action

1. Least action principle: variant 1

Consider a charge e located in a volume element dV moving with velocity $\mathbf{v}$. According the Lorentz-Wiechert force Law[1], this charge will experience a force from electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{1.1}$$

We now chose units such that $c = 1$.

Following Maxwell, we may introduce the scalar potential function Φ and the vector potential $\mathbf{A}$ with which the fields $\mathbf{E}$ and $\mathbf{B}$ can be expressed as follows:

$$\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{1.2}$$

and for the mechanical force, one then gets:

$$\mathbf{F} = -\nabla\Phi - \dot{\mathbf{A}} - \mathbf{v} \times (\nabla \times \mathbf{A}). \tag{1.3}$$

If one now defines a function L:

$$L \equiv \Phi - \mathbf{v} \cdot \mathbf{A}, \tag{1.4}$$

and introduces the total derivative, the force can be written in the form:

$$F_x = -\frac{dL}{dx} + \frac{d}{dt}\frac{\partial L}{\partial v_x}.$$

In so far as this has the form of a Lagrangian variation, one recognizes the validity of the following statement, which takes the first form of the Principle of least Action in Electrodynamics:

Proposition. *In any given electromagnetic field, a charged particle moves such that the integral*

$$\int dt(-T + \sum eL) \tag{1.5}$$

between two fixed times, given the start and end points, vanishes.

Herein the sum is understood to include all electric charges e, $-T$ is the kinetic energy of the masses carrying the charges e, and L is the quantity: $\Phi - \mathbf{v} \cdot \mathbf{A}$, which we designate the "electromagnetic potential."

A very simple expression for the electromagnetic potential can be obtained by exploiting the connection between the scalar and vector potentials with the propagation of electromagnetic waves. That is, if one lets $\rho(t', r')$ denote the charge density in the volume dV at time t, then at the location r' at time t', the following hold:

$$\Phi = \int dv \frac{\rho(t', r')}{r}, \quad \mathbf{A} = \int dv \frac{\rho(t', r')\mathbf{v}(t', r')}{r}. \tag{1.6}$$

Here r is the separation of the element dv from the source point r′; and, in each case the values of the charge density and velocity are to be those obtaining at a previous time corresponding to the time taken by light to traverse the separation. Using Eq. (1.6), the electronic potential takes the form:

$$L = \int dv \frac{\rho(t', r)}{r}[1 - \mathbf{v}'(t')\mathbf{v}(t', r)\cos(v, v')]. \tag{1.7}$$

[1] See, in particular: E. Wiechert, Electrodynamische Elementargesetze, *Drude's Annalen*, **Bd. 4**, 676-677.

This expression agrees with that of CLAUSIUS, except that here we use the retarded charge current and velocity. It represents the introduction of the delay of light transmission into action-at-a-distance. It admits a compact statement of the whole of electrodynamics: *All of electrodynamics can be melded with mechanics by introducing the term* $\sum eL$ *into the integrand of Hamilton's Principle.*

2. LEAST ACTION PRINCIPLE: VARIANT 2

While in the former variant all interaction is among the charge carriers, it would be useful for the purposes of optics to have a variant that determined electric and magnetic fields also. LORENTZ actually has proposed just such a principle in the foundations of his formulation of electrodynamics.[2] In addition, HELMHOLTZ formulated another version to extract the Hertz formulation of electrodynamics, which uses a novel variational principle.[3] Here we propose a variant that yields MAXWELL's equations, based on HELMHOLTZ's variational principle.

To include electromagnetic fields in Hamilton's principle, the following term:

$$\int dV dt \left(\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} + \rho L \right) \tag{2.1}$$

is put into the integrand.

It is understood that $\mathbf{E}$, $\mathbf{B}$ and L are to be expressed using Eqs. (1.2) and (1.4) with Φ and $\mathbf{A}$, and then the last four quantities are to be varied; in addition, the motion of the masses including their charges are also to be varied. This integral, as always with Hamilton's principle, is to be taken between two fixed times, and the variation of the orbits of the masses as well as the components of $\mathbf{A}$ must vanish at the two fixed times.

Let me indicate how the variation proceeds. The variation of the integral of Φ gives:

$$\mathbf{B} \cdot \nabla \delta\Phi + 4\pi\rho\delta\Phi.$$

When the partial integrations are executed, it follows from the condition that it vanish, that:

$$4\pi\rho - \nabla \cdot \mathbf{E} = 0.$$

The variation of $\mathbf{A}$ yields for the integrand:

$$\mathbf{B} \times \nabla\delta\mathbf{A} + \mathbf{E} \cdot \frac{\partial \delta\mathbf{A}}{\partial t} - 4\pi\rho\mathbf{v} \cdot (\delta\mathbf{A})$$

and thereby the conditional equation:

$$\nabla \times \mathbf{B} - \dot{\mathbf{E}} - 4\pi\rho\mathbf{v} = 0.$$

These constitute the first two MAXWELL equations. The other two are already contained in Eq. (1.2), that is, in the assertion that the six components of the electric and magnetic forces can be derived from the four quantities Φ and $\mathbf{A}$. In fact, simply by differentiating Eq. (1.2) one gets:

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}.$$

Concerning the variation of the orbits of the masses with their charges, there are two procedures, corresponding to the LAGRANGE and EULER formulations of hydrodynamics. Either one sets $dV\rho = de$ and follows the individual particles with their fixed charges de, which is the method used above; or one considers the variation of the charge density ρ and

[2] Lorentz, *La Théorie de Maxwell.*

[3] Helmholtz, *Gesammelte Abhandl.* Bnd. **III**, p. 476.

velocity which transpires for the whole motion at a particular place dV. I shall proceed along the latter route, although it is the more involved.

The coordinates of a massive point particle, which at time t has the coordinates $\mathbf{x}$, under variation at the same time has the position $\mathbf{x}+\vec{\xi}$. Likewise, the charge density ρ becomes $\rho+\delta\rho$. Corresponding to the continuity equation, one has:

$$\delta\rho = -\nabla(\rho\vec{\xi}).$$

The velocity of a particular particle changes under the variation by:

$$\frac{d\vec{\xi}}{dt} = \frac{\partial\vec{\xi}}{\partial t} + (\mathbf{v}\cdot\nabla)\vec{\xi}.$$

In order to get the change in velocity at the same location, one must deduct the amount

$$(\vec{\xi}\cdot\nabla)\mathbf{v}.$$

Thus, the total variation of $\mathbf{v}$, is:

$$\delta\mathbf{v} = \frac{\partial\vec{\xi}}{\partial t} + (\mathbf{v}\cdot\nabla)\vec{\xi} - (\vec{\xi}\cdot\nabla)\mathbf{v}.$$

If this variation is put into σL of Eq. (2.1), one obtains:

$$\sum\left[-\nabla(\rho\xi_i)(\Phi-\mathbf{v}\cdot\mathbf{A}) - \rho A_i\left(\frac{\partial\xi_i}{\partial t} + (\mathbf{v}\cdot\nabla)\xi_i\right) + \rho\xi_i\mathbf{A}(\frac{\partial\mathbf{v}}{\partial x_i})\right]$$

and this give for the force :

$$\rho\left[\nabla\Phi - (\mathbf{v}\cdot\mathbf{A}) + \frac{\partial}{\partial t}(\rho\mathbf{A})\right] + \sum\left[\nabla(\mathbf{v}A_i) + \rho\mathbf{A}\frac{\partial\mathbf{v}}{\partial x_i}\right],$$

or, when one takes into account, that during motion, the following equation naturally holds:

$$\dot{\rho} + (\nabla\cdot\mathbf{v})\rho = 0,$$

then with some slight simplifications, one gets:

$$\mathbf{F} = \rho[\nabla\Phi + \dot{\mathbf{A}} + \mathbf{v}\times(\nabla\times\mathbf{A})].$$

As this force is to be understood as that exercised on exterior charges, it can be seen to be in accord with Eqs. (1.1) and (1.3) from the theory of electrodynamics.

Thus, the whole of the Lorentz-Wiechert electrodynamics can be derived from a variational principle.

Part 2. **Elementary electrodynamic forces**

3. INTRODUCTION

Whereas in the preceeding note, the most compact form of Electrodynamics in a single formula was the goal, in this note the opposite tact is taken and without consideration for the complexity of the resulting expressions, an explicit form for the force exercised by a charged particle in arbitrary motion on another charge is sought.

For this purpose, only a small further step beyond WIECHERT's investigations in: "On the laws of electrodynamics"[4] need be taken.

[4] E. Wiechert, *Ann. der Phy.* **4**(2).

The total force, which a particle with charge e and velocity v experiences in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is given by the LORENTZ-WIECHERT formula:

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3.1}$$

where the velocity of light is set equal to 1; i.e., $c = 1$.

The electric and magnetic fields can be expressed in terms of the scalar potential Φ and the vector potential $\mathbf{A}$ in the form:

$$\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}; \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{3.2}$$

WIECHERT introduces (p. 682) for the potentials Φ and $\mathbf{A}$ originating from a charged particle moving with velocity v carrying charge e at the location of another particle with charge e' separated from e by the distance r the following expressions:

$$\Phi = e\left(\frac{1}{r(1 - v\cos(v,r))}\right)_{t=t'-r}; \quad \mathbf{B} = e\left(\frac{\mathbf{v}}{\mathbf{r}(1 - v\cos(v,r))}\right)_{t=t'-r}. \tag{3.3}$$

The distance r between e and e' is to be understood as the distance the source had in the past corresponding to the time is takes for light to traverse r. Thus, the expression in the parentheses are to be evaluated at the time $t = t' - r$ that meets this stipulation.

Should one seek to calculate using Eqs. (3.1) and (3.3) the force due to e on e', in principle it would be necessary only to differentiate with respect to the coordinates for e' after a time t'. This, however is not simple because the relative position of e' to e and the time of flight and velocity are changing. This problem can be handled in the following way:

The coordinates of e', which are considered independent with respect to differentiation, are x', y', z', the coordinates of e at an arbitrary time, are $x(t), y(t), z(t)$. Therefore:

$$r^2 = (x' - x(t))^2 + (y' - y(t))^2 + (z' - z(t))^2, \tag{3.4}$$

and only positions of e' for consideration are those for which:

$$t = t' - r. \tag{3.5}$$

When the earlier motion of e, as is being considered, is known such that x, y, z are given functions of t, then they may be taken as preconditions for t which can be expressed in terms of t', x', y', z'.

I now assert the following: all functions of x', y', z', t', t are to be given an overbar when they are considered to be dependant on these five variables. The overbar is to be dropped when in their original form t is eliminated using the condition of Eq. (3.5) so that they become functions of x', y', z', t. For the calculations implied in Eqs. (3.1) and (3.2), naturally the latter form is intended, that is, without overbar.

Then for an arbitrary function f, the following holds:

$$\frac{\partial f}{\partial x'} = \frac{\partial \bar{f}}{\partial x'} + \frac{\partial \bar{f}}{\partial t}\frac{\partial t}{\partial x'}; \quad \frac{\partial f}{\partial t} = \frac{\partial \bar{f}}{\partial t} + \frac{\partial \bar{f}}{\partial t}\frac{\partial t}{\partial t'},$$

and from Eq. (3.5), one gets:

$$\frac{\partial t}{\partial x'} = \frac{\frac{\partial \bar{r}}{\partial x'}}{1 + \frac{\partial \bar{r}}{\partial t}}; \quad \frac{\partial t}{\partial t'} = \frac{1}{1 + \frac{\partial \bar{r}}{\partial t}}. \tag{3.6}$$

Further, let us write Φ and $\mathbf{A}$ in the form:

$$\Phi = \frac{e}{\bar{r}\left(1 + \frac{\partial \bar{r}}{\partial t}\right)}; \quad \mathbf{A} = \dot{x}\Phi. \tag{3.7}$$

Also noteworthy, are the following expressions:

$$\frac{1}{2}\frac{\partial \bar{r}^2}{\partial x'} = x' - \bar{x}(t); \quad \frac{1}{2}\frac{\partial^2(\bar{r}^2)}{\partial x' \partial t} = -\frac{\partial \bar{x}}{\partial t}. \tag{3.8}$$

4. ELECTRIC FORCE

The electric force at a point x', y', z' and time t' , is given by:

$$\mathbf{F} = -\nabla' \Phi - \frac{\partial \mathbf{A}}{\partial t'}.$$

Using the notation introduced above, gives:

$$-K_x = \frac{\partial t}{\partial t'}\left\{\frac{\partial\left(\frac{\partial \bar{x}}{\partial t}\cdot\Phi\right)}{\partial t} + \frac{1}{\frac{\partial t}{\partial t'}}\left(\frac{\partial \bar{\Phi}}{\partial x'} + \frac{\partial \bar{\Phi}}{\partial t}\frac{\partial t}{\partial x'}\right)\right\}, \tag{4.1}$$

which with Eqs. (3.6, 3.7 and 3.8):

$$K_x = -\frac{1}{1+\frac{\partial \bar{r}}{\partial t}} Q_x,$$

$$Q_x = -\frac{\partial}{\partial t}\left(\frac{\bar{\Phi}}{2}\cdot\frac{\partial^2 \bar{r}^2}{\partial x' \partial t}\right) + \left(1 + \frac{\partial \bar{r}}{\partial t}\right)\frac{\partial \bar{\Phi}}{\partial x'} - \frac{\partial \bar{r}}{\partial x'}\frac{\partial \bar{\Phi}}{\partial t},$$

which can be rewritten as:

$$Q_x = \frac{\partial}{\partial x'}\left\{\bar{\Phi}\left(1 + \frac{\partial \bar{r}}{\partial t}\right)\right\} - \frac{\partial}{\partial t}\left\{\bar{\Phi}\left(\frac{\partial \bar{r}}{\partial x'} + \frac{1}{2}\frac{\partial^2 \bar{r}^2}{\partial x' \partial t}\right)\right\}.$$

In so far as

$$\frac{1}{\Phi} = \bar{r} + \frac{1}{2}\frac{\partial^2 \bar{r}^2}{\partial t}, \quad -\frac{1}{\Phi^2}\frac{\partial \bar{\Phi}}{\partial x'} = \frac{\partial \bar{r}}{\partial x'}\frac{1}{2}\frac{\partial^2 \bar{r}^2}{\partial x' \partial t},$$

it follows easily that:

$$E_x = -\frac{1}{1+\frac{\partial \bar{r}}{\partial t}}\frac{\partial P}{\partial x'}, \tag{4.2}$$

where

$$P = \frac{1}{\bar{r}} + \frac{1}{\Phi}\frac{\partial \bar{\Phi}}{\partial t} = \frac{1}{\bar{r}}\frac{1 - \frac{\partial^2 \bar{r}^2}{2\partial t^2}}{1 + \frac{\partial \bar{r}}{\partial t}}. \tag{4.3}$$

That is: Electric fields are derived from a kind of potential. If $\partial r/\partial t$ is expressed directly in terms of velocity **v** and acceleration, **a**, one gets:

$$P = \frac{1 - v^2 + ra\cos(ra)}{r(1 - v\cos(rv))}, \tag{4.4}$$

from which by differentiation by x', it follows:

$$E_x = -\frac{a_x}{r[1 - v\cos(rv)]^2} + \frac{1 - v^2 + ra\cos(ra)}{r^2[1 - v\cos(rv)]^2}\left(\frac{x' - x}{r} - v_x\right).$$

From this expression, one deduces that: *an electric field engendered by charge e at the point* (x', y', z') *comprises three components:*

(1) a force in the direction of the line joining the charges

$$\mathbf{E}_1 = \frac{ee'}{r^2}\cdot\frac{1 - v^2 + rv\cos(ra)}{r^2[1 - v\cos(rv)]^2};$$

(2) and of a force in the direction of the velocity v of charge e

$$\mathbf{E}_2 = -\mathbf{v}\cdot\mathbf{E}_1;$$

(3) and, a force in the direction of the acceleration, a of charge e:

$$\mathbf{E}_3 = -\frac{ee'}{r^2}\cdot\frac{a}{[1-v\cos(rv)]^2}.$$

Of course, it is to be understood that v and a were the velocity and acceleration of the charge e at the earlier moment when the fields were transmitted.

5. Calculation of the Magnetic and Mechanical Forces

Although it was rather complicated to calculate the electric field, once it has been obtained, the calculation of the magnetic field and mechanical force is easy.

To begin, one finds for the magnetic field

$$B_z = \frac{\partial A_x}{\partial y'} - \frac{\partial A_y}{\partial x'} = \frac{\partial}{\partial y'}\left(\frac{\partial x}{\partial t}\Phi\right) - \frac{\partial}{\partial x'}\left(\frac{\partial y}{\partial t}\Phi\right)$$
$$= \frac{\partial\bar{x}}{\partial t}\frac{\partial\bar{\Phi}}{\partial y'} - \frac{\partial\bar{y}}{\partial t}\frac{\bar{\Phi}}{\partial x'} + \frac{\partial t}{\partial y'}\frac{\partial}{\partial t}\left(\frac{\partial\bar{x}}{\partial t}\bar{\Phi}\right) - \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}\left(\frac{\partial\bar{y}}{\partial t}\bar{\Phi}\right)$$

Taking into account the formulas just developed, namely:

$$-\frac{1}{\Phi^2}\frac{\partial\bar{\Phi}}{\partial x'} = \frac{\partial\bar{r}}{\partial x'}\frac{1}{2}\frac{\partial^2\bar{r}^2}{\partial x'\partial t} = \frac{\partial\bar{r}}{\partial x'} - \frac{\partial\bar{x}}{\partial t},$$

$$\frac{\partial t}{\partial t'} = -\frac{\frac{\partial\bar{r}}{\partial x'}}{1+\frac{\partial\bar{r}}{\partial t}} = -\frac{\partial\bar{r}}{\partial x'}\frac{\partial t}{\partial t'}$$

as well as those for the other coordinates, one obtains:

$$B_z = \frac{\partial\bar{r}}{\partial x'}\left\{\frac{\partial\bar{\Phi}}{\partial y'} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\left(\frac{\partial\bar{y}}{\partial t}\bar{\Phi}\right)\right\} - \frac{\partial\bar{r}}{\partial y'}\left\{\frac{\partial\bar{\Phi}}{\partial x'} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}\left(\frac{\partial\bar{x}}{\partial t}\Phi'\right)\right\}$$

or, in view of Eq. (4.1), the electric force

$$B_z = \frac{\partial\bar{r}}{\partial x'}\left\{-E_y - \frac{\partial\bar{\Phi}}{\partial t}\frac{\partial\bar{r}}{\partial y'}\frac{\partial t}{\partial t'}\right\} - \frac{\partial\bar{r}}{\partial y'}\left\{-E_z - \frac{\partial\bar{\Phi}}{\partial t}\frac{\partial\bar{r}}{\partial z'}\frac{\partial t}{\partial t'}\right\},$$

so that, finally:

$$B_z = \frac{\partial\bar{r}}{\partial y'}E_x - \frac{\partial\bar{r}}{\partial x'}E_y. \tag{5.1}$$

As the differential quotients, $\partial\bar{r}/\partial x'$, etc. are simply direction cosines of r, this final result may be expressed in words as follows:

The Magnetic force is perpendicular to the radius vector and the electric force and has the magnitude:

$$B = E\sin(E,r).$$

The mechanical force experienced by charge e' moving with velocity v', according to Eq. (3.1), is comprised of an electric component with an additional component, which is perpendicular to the magnetic force and v' and has the magnitude $\mathbf{B}\mathbf{v}'\sin(\mathbf{B},v')$. Simple geometric considerations regarding the relationship between E and B lead to conclusion that it is in the plane of (E,r) and has the magnitude

$$\mathbf{E}v'\sin(\mathbf{E},r)\cos(v',[\mathbf{E},r]).$$

In words this is expressed as follows: *The mechanical force exercised by charge e on charge e' moving with velocity v', comprised the electric force E that e generates at the location of e', and an auxiliary force. This auxiliary force is in the plane specified by the direction of the electric force and the line r joining e' with e, and perpendicular to v'. It has the magnitude*

$$\mathbf{E}u' \sin(\mathbf{E}, r),$$

where u' is the projection of v' onto this plane. (Of the two possible directions for the auxiliary force, the one in rotational sense of $(\mathbf{B}, r)$ from u' is to be chosen.

It is in principle an interesting result that the elementary forces between point charges conforms with mechanics in the sense that it depends on first and second order derivatives with respect to time. However, it involves quantities of this character for two different times, namely the velocity and acceleration of the source charge at times of signal generation whereas for the affected charge at times of impact. Thus, the equations of motion for a system of point charges, while being second order differential equations, are complicated by the fact that they are also simultaneously functional differential equations.

6. Alternate forms of mechanical force

Putting the value for the magnetic force, Eq. (5.1), into the expression for mechanical force, Eq. (3.1) (in which a prime indicates that it is to be understood that the effect in on the charge e'), and then uses Eqs. (4.2) and (4.3), gives as an expression for the mechanical force

$$\mathbf{F} = -\frac{1 - v' \cos(rv')}{1 + \frac{\partial \bar{r}}{\partial t}} \cdot \frac{\partial P}{\partial x'} + \frac{\frac{\partial \bar{r}}{\partial x'}}{1 + \frac{\partial \bar{r}}{\partial t}} \left(\mathbf{v}' \cdot \nabla P\right).$$

One introduces now as new variables the coordinates of the point at which e' was located as the wave departed e, as if it the whole of the intervening time had had the velocity v'. These are then given by

$$\begin{aligned} \mathfrak{x} &= x' - v'_x \bar{r}; \\ \mathfrak{y} &= y' - v'_y \bar{r}; \\ \mathfrak{z} &= z' - v'_z, \end{aligned}$$

where $\bar{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Thus, for an arbitrary function $f(x', y', z')$:

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial \mathfrak{x}} - \frac{\partial \bar{r}}{\partial x_1} \left(\mathbf{v} \cdot \nabla_{\mathfrak{r}} f\right)$$

(where $\nabla_{\mathfrak{r}}$ is the gradient with respect to $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$)

$$\mathbf{v}' \cdot \nabla f = (1 - v' \cos(rv')\,(\mathbf{v} \cdot \nabla f)\,.$$

Applied to F_x, these give:

$$F_x = -\frac{1 - v' \cos(rv')}{1 - v \cos(rv)} \cdot \frac{\partial P}{\partial \mathfrak{x}}, \tag{6.1}$$

according to which *the complete mechanical force after removal of a certain factor, can be obtained by differentiation of a potential P with respect to the new variables* $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$.

7. Case I: Two identical charges in equal parallel motion

In view of Eq. (6.1), when $v = v'$, then:

$$F_x = -\frac{\partial P}{\partial \mathfrak{x}}.$$

Obviously, in this case, force can be derived from a potential. If we let the velocities of the two charges be parallel to the x axis, then the potential, according to Eq. (4.4), would be:

$$P = \frac{1-v^2}{r - v(x'-x)}.$$

Introducing the variables $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$, in place of x', y', z' gives with a simple calculation:

$$P = \frac{1-v^2}{\sqrt{(\mathfrak{x}-x)^2 + [(\mathfrak{y}-y)^2 + (\mathfrak{z}-z)^2](1-v^2)}}.$$

Thus, one gets familiar results for the force between two comoving charges.

8. Case II: Force of uniformly moving charge on a free charge

If the acceleration a of the source charge e vanishes, the potential becomes

$$P = \frac{1-v^2}{r - \mathbf{v}\cdot(\mathbf{r}'-\mathbf{r})}$$

and it follows from §2 that the force is:

$$E_x = \frac{1-v^2}{r^2[1 - v\cos(rv)]^2}[x' - x - v_x r].$$

Let:

$$\begin{gathered}\xi = x' - x - v_x r;\\ \eta = y' - y - v_y r;\\ \zeta = z' - z - v_z r;\\ r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2;\\ \rho^2 = \xi^2 + \eta^2 + \zeta^2,\end{gathered}$$

then ξ, η, ζ are the coordinates of e' relative to the simultaneous position of e, ρ is the real separation of the charges, both valid at the moment of impact of the wave on e'. If one eliminates x', y', z' with the help of ξ, η, ζ, one gets:

$$E_x = \frac{1-v^2}{[1 - v^2\sin^2(v\rho)]^{3/2}} \cdot \frac{\xi}{\rho^2}.$$

That is: *The electric force is directed toward the* ***simultaneous*** *position of the source charge and has the magnitude:*

$$E = \frac{1}{\rho^2} \cdot \frac{1-v^2}{[1 - v^2\sin^2(v\rho)]^{3/2}}.$$

To find the auxiliary mechanical force, note that, the triangle formed by e and e' at the moment of impact of the wave and the position of the source e at the moment of the launch of the wave has the sides r, ρ and $\mathbf{r}\cdot\mathbf{v}$ so that

$$r\sin(r,\rho) = r\sin(r,\mathbf{E}) = r\cdot v\sin(v,\rho),$$

or

$$\sin(r,\mathbf{E}) = v\sin(v,\rho).$$

With this and §3, it follows that the auxiliary force lies in the plane defined by the lines joining e, e' and v of e, perpendicular to the velocity v' of e' with the magnitude:

$$\frac{1-v^2}{[1-v^2\sin^2(v\rho)]^{3/2}} \cdot \frac{v \cdot v'}{\rho^2} \sin(v,\rho)\cos(v'[\rho,v]).$$

This result reminds one of GRASSMANN's Law of the older form of electrodynamics, so that one can in fact express this result in words so:

The mechanical force exercised by uniformly moving charge on a free charge, can be obtained from the sum of COULOMB*'s Law and* GRASSMANN*'s Law multiplied by the correction factor:*

$$\frac{1-v^2}{[1-v^2\sin^2(v\rho)]^{3/2}},$$

where v is the velocity of the source charge, and ρ *is the displacement between them.*[5]

It must be noted, however, that although this result applies to the motion of individual charges, for current in closed circuits, curvature causes accelerations that can not be ignored. Thus, one may not infer that the correction factor can be used for closed circuits.

9. CASE III: FIELDS AT A GREAT DISTANCE FROM THEIR SOURCE

The potential P from which the force in §2 was derived, according to Eq. (4.4), has the magnitude:

$$P = \frac{1-v^2}{r(1-v\cos(rv))} + \frac{a\cos(ra)}{1-v\cos(rv)}.$$

For sufficiently large distances from the source charge, even when a does not zero, the first term vanishes, so that

$$P = \frac{a\cos(ra)}{1-v\cos(rv)}.$$

In so far as this then depends on the direction but not the size of r, it follows that:

$$\frac{\partial P}{\partial r} = 0,$$

that is, *at large distances from the source charge the electric force is perpendicular to the radius vector. Further, from §3, it is seen, that at large distances the magnetic force equals the electric force, and is perpendicular to both the electric force and the radius vector.*

Thus, at large distances from the source, (except for vanishing acceleration), conditions prevail that are familiar from the theory of light waves.

When the velocity v is small compared to the speed of light, then to good approximation the magnitude of the electric and magnetic forces equals:

$$\mathbf{E} = \mathbf{B} = \frac{a\sin(ar)}{r}.$$

From this one sees, as already often implied, that from the intensity of light generated by an oscillating charge, depends not on the source's amplitude or velocity, rather on its acceleration.

[5]In *The Encyclopedia of the Mathematical Sciences, Vol V,* p. 12 by REIF and SOMMERFELD, the latter emphasizes the relationship between electronic theory and the older CLAUSIUSian Potential Law as well as with GRASSMANN'S Law. This relationship is confirmed and elaborated by §1 of the preceeding paper and §3 and §6 of this paper.

Part 3. **Electron motion**

10. INTRODUCTION

ABRAHAM's theory of a massless, purely translating electron has been verified to an astounding degree by KAUFMANN.[6] On the other hand, experiments thus far have revealed no indication that an electron exhibits rotation. This, however, seems absurd. It is unbelievable that electrons departing a cathode with enormous speed never rotate. Thus, the situation must be, that whatever effect rotation has, it is not evident in terms of its visible motion. The question then: is it possible to use this, negative, fact in the LORENTZ-WIECHERT formulation of electrodynamics of massless electrons? The first impression is doubtful. In the two simplest cases, of rotation without translation and rotation about the direction of motion, which were treated in ABRAHAM's work, the effect of rotation was small. As soon, however, as the rotation axis was at an angle to the direction of motion, as ABRAHAM implied, it seems that even without external forces, such an electron no longer moves in a straight line, but executes complex maneuvers. This is a matter that must be settled before the notion of a massless electron can be considered as logically established.

If one now bases his considerations on a variational principle, as in the preceeding articles, it turns out that both ABRAHAM's results can be obtained relatively easily and in addition, extended to the case of arbitrary motion and arbitrary rotation, which leads to a full theoretical treatment of the question posed above. Naturally the assumption of small accelerations or "quasistationary" motion is to be maintained. This implies, as will be shown, that rotation is restricted such that the velocity of the surface of an electron is substantially below that of light.

The following study has its own intrinsic interest, in so far as it reveals the structure of variational principles as much as it concerns the motion of electrons. It will shown, that the motion of an electron follows from very general considerations from variational techniques in which functions of all orders of differential quotients of the coordinates arise. In the special case of quasistatic motion, the variational principle effectively becomes a minimalization principle, to be distinguished from HAMILTON's Principle only in that the potential depends not only on velocity, but also acceleration and in that for the limits both the coordinates and velocities are to be held constant. That is, one has to do here with an example of KÖNIGSBURGER's extended mechanics. By restriction to ABRAHAM's "selected motion," the accelerations disappear from the potential and one arrives at a very close analog to conventional mechanics.

11. ELECTRON MOTION AND RETARDED POTENTIALS

Let ρ be the charge density in a spacial infinitesimal volume dV, for which $\mathbf{v}$ is the velocity. Also, let the velocity of light $c = 1$, even though it should be visible in formulas to make units obviously correct. One now defines the retarded potentials Φ and $\mathbf{A}$ via the integrals

$$\Phi(t,\mathbf{x}^0) = \int \frac{dV}{r}[\rho]_{t-\frac{r}{c}}, \quad \mathbf{A}_{\mathbf{v}}(t,\mathbf{x}^0) = \int \frac{dV}{r}[\rho\mathbf{x}]_{t-\frac{r}{c}}. \tag{11.1}$$

where, r is the distance of dV from point of interest $\mathbf{x}^0$. The subscript $t - r/c$ means, that in each volume element dV, that value of ρ or $\rho\mathbf{v}$ is to be used that corresponds to that time preceeding t to account for the time-of-flight of the signals from source to point of interest.

[6] Kaufmann, *Ann. der Phys.* **10** 105 (March 1903); *Gött. Nach.* **5** (3).

We now examine the special case for which the following holds: We consider a single electron in space, comprised of a sphere of radius R within which charge is uniformly distributed. Outside the electron the charge density is considered to be null everywhere. The coordinates and velocity of the center of the sphere are **a** and **u** respectively, its axis of rotation is taken to be about the vector **p**. Further, one takes:

$$x = a_x + \xi, \quad y = a_y + \eta \quad z = a_z + \zeta,$$

where $\vec{\xi}$, i.e., ξ, η, ζ, are relative coordinates with origin at the center of the sphere. Thus, the velocity of an arbitrary point of the electron are[7]

$$\mathbf{v} = \mathbf{u} + \mathbf{p} \times \vec{\xi}.$$

It is taken that the motion of the electron is analytic; that is, that the coordinates of the center, the rotational velocity for each value of t and small enough values of $t' - t$ all can be expressed as in terms of a convergent series:

$$a_i(t') = a_i(t) + u_i(t)(t' - t) + u_i' \frac{(t' - t)^2}{2!} + \cdots,$$

$$p_i(t') = p_i(t) + p_i'(t' - t) + p_i'' \frac{(t' - t)^2}{2!} + \cdots, \tag{11.2}$$

where $i = x, y, z$.

Under these conditions, it is possible to expand both Φ and **A** for an arbitrary time t as power series of u_i', u_i'', p_i', p_i'', etc. This is most clearly achieved by developing the first few terms of the acceleration u_i and p_i, especially as we shall need them eventually in any case.

To simplify the notation, let $x, y, z = x_i$; $i = x, y, z$ and $\xi, \eta, \zeta = \xi_i$, and let summations be understood as over i so that the dependence of a function on all three dimensions is implied even when only one is written.

Beginning with the calculation of Φ, consider that in each infinitesimal volume element there is a charge density ρ at time $t - r$ (for $c = 1$), which is a function of

$$\xi_i = x_i - a_i(t - r),$$

such that it is constant for $\sum \xi_i^2 < R^2$ and zero otherwise. The full expression of Φ is

$$\Phi(x_i^0, t) = \int \frac{dxdydz}{\sqrt{\sum (x_i^0 - x_i)^2}} \rho[x_i - a_i(t - r)].$$

It is now advantageous to introduce the relative coordinates, ξ_i. Using the Jacobian of the transformation from x_i to ξ_i gives:

$$\Phi(x_i^0, t) = \int \frac{d\xi d\eta d\zeta \rho(\xi, \eta, \zeta)}{N},$$

where $N = r - (\mathbf{x}^0 - \mathbf{x}) \cdot \mathbf{u}(t - r)$. It remains to determine N, for which, it will turn out, that this is exactly the point where the analyticity of the motion plays a critical role. From:

$$r^2 = |(\mathbf{x}^0 - \mathbf{x})|^2 = \sum [x_i - \xi_i - a_i(t - r)]^2,$$

with the power series expansion of **a**, one gets:

$$r^2 = \sum \left[x_i^0 - \xi_i - a_i(0) + u_i(t) r - u_i'(t) \frac{r^2}{2} + u_i''(t) \frac{r^3}{3!} + \cdots \right]^2. \tag{11.3}$$

[7] *Translator's note:* KS did not use GIBBSian vector notation as employed herein for convenience.

From this, generally transcendental, equation one can obtain a series expression in u_i', u_i'', etc., for r in terms of ξ.

One can obtain a first approximation of $\bar{r}$ by ignoring u_i', u_i''. The fundamental equation for $\bar{r}$ then becomes a quadratic for which the positive solution is

$$\bar{r}(1-u^2) = k + l, \tag{11.4}$$

where

$$\begin{gathered} u^2 = \sum u_i^2, \\ \xi_i^0 = x_i^0 - a_i(t), \\ k = \sqrt{(1-u^2)\sum(\xi_i^0 - \xi_i)^2 + l^2}, \\ l = \sum u_i(\xi_i^0 - \xi_i). \end{gathered} \tag{11.5}$$

The ξ_i^0 are relative coordinates of the sub-charge at the point of interest, (Aufpunkt), with respect to the center of charge of the electron at time t. The root k is always to be taken as an absolute value.

Putt these approximations into the elements with u_i' in the basic equation one obtains with respect to the first power of u_i' the second order approximation, which is where we shall cease:

$$r = \bar{r} - \frac{\bar{r}^2}{2k}\sum u_i u_i' - \frac{\bar{r}^2}{2k}\sum(\xi_i^0 - \xi_i)u_i'. \tag{11.6}$$

Thus, the whole denominator N:

$$\begin{aligned} N &= r - \sum[\xi_i^0 + a_i(t) - \xi_i - a_i(t-r)]u_i(t-r) \\ &= r - \sum[\xi_i^0 - \xi_i + u_i(t)r - u_i'(t)\frac{r^2}{2} + \cdots][u_i - u_i'r + \frac{u_i''}{2}r^2\cdots], \end{aligned}$$

from which with the approximations Eqs. (11.4) and (11.6) to the specified degree, gives:

$$N = k + \frac{k^2 - l^2}{2k(1-u^2)}\cdot\sum u_i'(\xi_i^0 - \xi_i) + \frac{2k^2 + 3k^2 l - l^3}{2k(1-u^2)}\sum u_i u_i'. \tag{11.7}$$

The development of A_i proceeds in an analogous manner. One needs only to introduce in the integrand the factor:

$$v_x(t-r) = u_x + p_y\zeta - p_z\eta - r[u_x' + p_y'\zeta - p_z'\eta] + \frac{r^2}{2}[u_x'' + p_y''\zeta - p_z''\eta]\cdots.$$

Let:

$$g_i(t, \mathbf{x}^0) = \int \frac{d\xi d\eta d\zeta \rho \xi_i}{N},$$

then we may write:

$$\begin{aligned} A_x &= u_x\Phi + p_y g_z - p_z g_y - \frac{u_x'}{1-u^2}\int \partial\xi\partial\eta\partial\zeta\rho\left[1 + \frac{l}{k}\right] \\ &\quad - \frac{p_y'}{1-u^2}\int \partial\xi\partial\eta\partial\zeta\rho\zeta\frac{1}{k} + \frac{p_z}{1-u^2}\int \partial\xi\partial\eta\partial\zeta\rho\eta\frac{1}{k} \\ &\quad + \frac{p_y''}{1-u^2}\int \partial\xi\partial\eta\partial\zeta\rho\zeta\frac{(l+k)^2}{2k} - \frac{p_z''}{1-u^2}\int \partial\xi\partial\eta\partial\zeta\rho\eta\frac{(l+k)^2}{2k}. \end{aligned} \tag{11.8}$$

In view of requirements for developments to follow, in this equation I include the terms with p_i'', etc.

It is obvious how this development starting from the fundamental equation, Eq. (11.3), can be carried further. The convergence issue for the resulting series can be resolved with the method of majorants. The convergence evidentially will depend whether the potential of an electron is to be evaluated at a near or far point, $\mathbf{x}^0$. The further this point is distant, the earlier the positions of the electron that come into consideration must be, and the larger the time spans $t - t'$ In Eq. (11.2) must be. Thus, the method of majorants gives the following convergence theorem:

The series expansion of the coordinates of the center of charge and the rotational velocities of an electron as a series in $(t - t')$ will converge for intervals satisfying $(t - t') < T$. The expansion of the retarded potential for time t at point $\mathbf{x}^0$, will converge when $\mathbf{x}^0$ is inside a sphere centered at the center of the electron (at time t) of radius

$$r < VTk,$$

where k is a finite factor when the electron's velocity is less than that of light.

Moreover, when $T > R/(kV)$, i.e., T is greater than a particular finite factor of the time, then the series inside the electron also converge.

The expansions Eqs. (11.7) and (11.8) for Φ and $\mathbf{A}$ and this convergence theorem are needed below.

12. A VARIATIONAL PRINCIPLE

The variational principle considered above can be expressed as follows. One considers the quantity:

$$2J = \int dV\rho[-\Phi + \mathbf{v}\cdot\mathbf{A}], \tag{12.1}$$

where the integration is over all space. Here Φ and $\mathbf{A}$ are the retarded values calculated from Eq. (11.1). Let $\mathbf{F}$ be an external force operating on the charge. One considers now, in addition to the actual motion of the charges between the times t_0 and t_1, a virtual displacement at $\mathbf{x}$ of $\delta\mathbf{x}$ at time t. The resulting alterations of ρ and $\rho\mathbf{v}$, are denoted by $\delta\rho$ and $\delta(\rho\mathbf{v})$. One then has

$$\int_{t_0}^{t_1}\int dV[-\delta\rho\cdot\Phi + \sum\delta(\rho v_i)A_i + \sum F_i\delta x_i] = 0,$$

where the δx_i vanish at the time t_0 and t_1. Were this restriction to be lifted, then the integral would not vanish, but instead equal

$$\int dV\rho\mathbf{x}\cdot\mathbf{A}|_{t_1}^{t_0}. \tag{12.2}$$

The peculiarity of this variational principle is, that the fields specified by Φ and $\mathbf{A}$ are not to be varied, but considered "frozen." Thus, for our purposes, this principle is not appropriate, because for the motion of an electron, these fields themselves depend on the this motion. That is, were we to put retarded potentials in the expression Eq. (11.1) for J, we would get a function of the electron's velocity in which these velocities may not be varied everywhere. That is, the variation would be incomplete.

This difficulty can be handled in the most direct way by simply, contrary to the above condition, doing a total variation on J in order to achieve a universal variation. This results in a new variational principle whose characteristics are appropriate for our purposes.

Let the anticipatory displacements of the motion of the charges that engender Φ and $\mathbf{A}$ at the position $\mathbf{x}$ be denoted $\delta\Phi$ and $\delta\mathbf{A}$. Then, for a total variation of J, one has

$$\delta\int 2Jdt = \int dt\int dV[-\delta\rho\Phi - \rho\delta\Phi + (\delta\rho\mathbf{v})\cdot\mathbf{A} + \rho\mathbf{v}\cdot\delta\mathbf{A}].$$

This variation can be reformulated as follows. The retarded potentials obviously satisfy the equations:

$$\frac{\partial^2\Phi}{\delta t^2} = \triangle\Phi + 4\pi\rho,$$
$$\frac{\partial^2\mathbf{A}}{\delta t^2} = \triangle\mathbf{A} + 4\pi\rho\mathbf{v}.$$

From this it follows:

$$\frac{\partial^2\delta\Phi}{\partial t^2} = \triangle\delta\Phi + 4\pi\rho,$$
$$\frac{\partial^2\delta\mathbf{A}}{\partial t^2} = \triangle\delta\mathbf{A} + 4\pi\delta(\rho\mathbf{v}).$$

Restricting the volume V of the integration to the interior of a surface s which is chosen so as to contain all the charges, and letting the normal to s be $\mathbf{n}$, then a direct application of GREEN's formula gives:

$$\begin{aligned}\delta\int_{t_0}^{t_1} 2Jdt = &\int_{t_0}^{t_1} dt\int dV[-2\delta\rho\Phi + 2\delta(\rho\mathbf{v}\cdot\mathbf{A})]\\ &+\int_{t_0}^{t_1} dt\int\frac{dV}{4\pi}\left[\Phi\frac{\partial^2\delta\Phi}{\partial t^2} - \delta\Phi\frac{\partial^2\Phi}{\partial t^2} - \mathbf{A}\cdot\frac{\partial^2\delta\mathbf{A}}{\partial t^2} - (\delta\mathbf{A})\cdot\frac{\partial^2\mathbf{A}}{\partial t^2}\right]\\ &+\int_{t_0}^{t_1} dt\int\frac{dV}{4\pi}\left[\delta\Phi\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial\delta\Phi}{\partial n} - (\delta\mathbf{A})\cdot\frac{\partial\mathbf{A}}{\partial n} - \mathbf{A}\cdot\frac{\partial\delta\mathbf{A}}{\partial n}\right].\end{aligned}$$

Concerning the first integral, we know from Eq. (12.1) and (12.2) that it has the value

$$2\left[\int dV\rho\mathbf{A}\cdot d\mathbf{x}\right]_{t_0}^{t_1} - 2\int_{t_0}^{t_1} dt\delta\mathbf{A},$$

where we use the abbreviation: $\delta\mathbf{A} = \int dV\mathbf{F}\cdot d\mathbf{x}$, i.e., the work done by external forces. Further, introducing the expression

$$2\delta U = \Phi\frac{\partial\delta\Phi}{\partial t} - \delta\Phi\frac{\partial\Phi}{\partial t} - \mathbf{A}\cdot\frac{\partial\delta\mathbf{A}}{\partial t} + (\delta\mathbf{A})\cdot\frac{\partial\mathbf{A}}{\partial t},$$

then the second integral becomes

$$\int_{t_0}^{t_1} dt\int\frac{dV}{2\pi}\frac{\partial\delta U}{\partial t}.$$

If we now would let $s\to\infty$ while holding all charges as they were, the surface integral would vanish such that

$$\int_{t_0}^{t_1} dt[\delta J + \delta A] = \left[\int dV\left(\frac{\delta U}{4\pi} + \rho\mathbf{A}\cdot\delta\mathbf{x}\right)\right]_{t_0}^{t_1}.$$

Nevertheless, this variation principle does not meet our needs here to use the series expressions developed above for Φ and $\mathbf{A}$, as these series converge only in a restricted region. Thus, here I shall chose a finite surface for the volume containing the electron—further, as here I return from the general case to considering only a single electron—a sphere centered on the charge with arbitrary radius.

It will be useful from now on to denote the point from which Φ and $\mathbf{A}$ depend, and which are covered by the integrations indicated above over V and s, by $\mathbf{x}^0$ as used above, which

in turn can be made evident with the notation: dV^0 and ds^0. Further, if one introduces the relative coordinates of the passive point with respect to the center of the electron, i.e.,

$$\vec{\xi} = \mathbf{x}^0 - \mathbf{a}^0$$

and denotes differentiation with respect to time while holding $\vec{\xi}$ fixed with d/dt, then

$$\frac{\partial}{\partial \xi_0^0} = \frac{\partial}{\partial x_0^0}, \quad \frac{\partial}{\partial t} = \frac{d}{dt} - \mathbf{u}^0 \cdot \frac{\partial}{\partial \mathbf{x}^0}. \tag{12.3}$$

Observing that V^0 in the relative coordinates is invariant, leads to

$$\int_{t_0}^{t_1} dt \int \frac{dV^0}{2\pi} \frac{\partial \delta U}{\partial t} = \left[\int \frac{dV^0}{2\pi} \delta U \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \int \frac{ds^0}{2\pi} \mathbf{u_n} \delta U,$$

where $\mathbf{u_n}$ is to be understood as the projection of the velocity of the electron on the normal ds^0.

Finally, by using the abbreviation

$$\frac{\partial}{\partial n} + u_n \frac{\partial}{\partial t} = \frac{\partial}{\partial \sigma}, \tag{12.4}$$

one obtains the following form for the variational principle:

$$\begin{gathered} \textstyle\int_{t_0}^{t_1} dt(\delta J + \delta R) = [\delta Y]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \delta W. \\ \delta W = \tfrac{1}{8\pi} \textstyle\int ds^0 \left[\delta\Phi \frac{\partial \Phi}{\partial \sigma} - \Phi \frac{\partial \delta\Phi}{\partial \sigma} - \right] - \delta\mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \frac{\partial \delta\mathbf{A}}{\partial t}]. \\ \delta Y = \textstyle\int dV^0 \rho \mathbf{A} \cdot \delta\mathbf{x} - \frac{1}{8\pi} \int dV_0 \left[\delta\Phi \frac{\partial \Phi}{\partial t} - \Phi \frac{\partial \delta\Phi}{\partial t} - \delta\mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \frac{\partial \delta\mathbf{A}}{\partial t} \right]. \end{gathered} \tag{12.5}$$

The infinitesimal work element δR which is due to external forces, for an individual electron has the value:

$$\delta R = \mathbf{S} \cdot \delta\mathbf{a} + \mathbf{D} \cdot \delta\vec{\omega},$$

where $\mathbf{S}$ and $\mathbf{D}$ are external translation and rotation forces, and $\delta\mathbf{a}$ and $\delta\vec{\omega}$, infinitesimal translations and rotations.

Were we to wish to consider electrons with mass—which is explicitly not our intention here—then it would be necessary to add the term $\delta \int dm\mathbf{v}^2/2$ to the integral on the left of the first of Eqs. (12.5).

13. General remarks on analytical variational principles

Reflecting the variational principle just developed expressed using the series developed in P 2 for Φ and $\mathbf{A}$, reveals the following characteristics: J is a series expansion in $\mathbf{u}', \mathbf{u}'', \mathbf{p}', \mathbf{p}''$, etc., for which the coefficients depend on $\mathbf{u}$ and $\mathbf{p}$. The quantitates δW and δV then take the form:

$$\begin{aligned} \delta W &= \delta\mathbf{a} \cdot \mathbf{R} + \delta\vec{\omega} \cdot \mathbf{C} + \delta\mathbf{u} \cdot \mathbf{R}_1 + \delta\mathbf{p} \cdot \mathbf{C}_1 + \delta\mathbf{u}' \cdot \mathbf{R}_2 + \delta\mathbf{p}' \cdot \mathbf{C}_2 + \cdots, \\ \delta Y &= \delta\mathbf{a} \cdot \mathbf{P} + \delta\vec{\omega} \cdot \mathbf{Q} + \delta\mathbf{u} \cdot \mathbf{P}_1 + \delta\mathbf{p} \cdot \mathbf{Q}_1 + \delta\mathbf{u}' \cdot \mathbf{P}_2 + \delta\mathbf{p}' \cdot \mathbf{Q}_2 + \cdots, \end{aligned} \tag{13.1}$$

where $\mathbf{R}, \mathbf{C}, \mathbf{P}, \mathbf{Q}, \mathbf{R}_1, \mathbf{P}_1$, etc. are all also series expansions in $\mathbf{u}', \mathbf{p}'$ etc.

If now the surface s is shrunk down to the surface of the electron, then the convergence theorem from §2, assures us of the convergence of the quantities $J, \mathbf{R}, \mathbf{P}$ etc. when for the m-th derivative of $\mathbf{u}$ and $\mathbf{p}$ the following holds:

$$\mathbf{u}^{(m)} < \frac{\gamma}{T^m}; \quad \mathbf{p}^{(m)} < \frac{\gamma}{T^m} \left(T = \frac{1}{k} \frac{R}{Y}, \, \gamma \text{ finite} \right) \tag{13.2}$$

In order for the whole series, Eq. (13.1), for δW and δY also converge, it is necessary that the varied trajectories also have the same analytic properties, i.e. which requires:

$$\frac{\partial \mathbf{u}^{(m)}}{\partial \alpha} < \frac{\gamma'}{T^m}; \quad \frac{\partial \mathbf{p}^m}{\partial \alpha} < \frac{\gamma'}{T^m} \ (\gamma' \text{ finite}), \tag{13.3}$$

if we make the trajectory dependant on a parameter α, and then effect the variation of the trajectory by varying α.

Regardless of how general and unconstrained the variational principle Eq. (12.5) might be, as soon as series expansions in terms of $\mathbf{u}', \mathbf{p}'$and their variations are used, only analytic trajectories with the limitations expressed by Eqs. (13.2) and (13.3) are available. Thus, one can not extract the usual differential equations of a variational procedure, because the values of the variables $\mathbf{a}, \mathbf{u}, \mathbf{u}'$ etc., for the beginning time already determine the development of the motion and a clean differentiation between the beginning, ending and intervening values is not possible. Nevertheless, the differential equations that one would obtain by the usual variational principle pertain in spite of the restriction to analytic trajectories.

One obtains from Eqs. (12.1) and (12.2) via displacement of upper limit of integration, t_1,

$$\delta J = +\delta R = \delta W + \frac{d\delta Y}{dt}. \tag{13.4}$$

Furthermore:

$$\begin{aligned} \delta W + \frac{d\delta Y}{dt} &= \delta \mathbf{a} \cdot \left(\frac{d\mathbf{P}}{\mathbf{dt}} + \mathbf{R} \right) + \delta \mathbf{u} \cdot \left(\frac{d\mathbf{P}_1}{dt} + \mathbf{P} + \mathbf{R}_1 \right) + \delta \mathbf{u}' \cdot \left(\frac{d\mathbf{P}_2}{dt} + \mathbf{P}_2 + \mathbf{R}_2 \right) \\ &+ \delta \vec{\omega} \cdot \left(\frac{d\mathbf{Q}}{\mathbf{dt}} + \mathbf{C} \right) + \frac{d\delta \mathbf{Q}}{dt} \cdot \mathbf{Q} + \delta \mathbf{p} \cdot \left(\frac{d\mathbf{Q}_1}{dt} + \mathbf{C}_1 \right) + \delta \mathbf{p}' \cdot \left(\frac{d\mathbf{Q}_2}{dt} + \mathbf{Q}_1 + \mathbf{C}_2 \right) + \cdots . \end{aligned} \tag{13.5}$$

At the moment t_1, all variations $\delta \mathbf{a}, \delta \mathbf{u}, \delta \mathbf{u}', \cdots \delta \vec{\omega}, \delta \mathbf{p}, \delta \mathbf{p}' \ldots$ can be, within certain boundaries determined by convergence considerations, arbitrary. Only the quantity $d\delta\vec{\omega}/dt$ is fixed by this procedure. Considering for the moment $\delta\vec{\omega}$ as a function of time, one may decompose an elementary rotation as a composition:

$$\delta \mathbf{p} = \frac{d\delta\vec{\omega}}{dt} - (\delta\vec{\omega}) \times \mathbf{p}$$

from which the sought expression for $d\delta\vec{\omega}/dt$ in terms of $\delta\vec{\omega}$ and $\delta\mathbf{p}$, is:

$$\frac{d\delta\vec{\omega}}{dt} = \delta \mathbf{p} + (\delta\vec{\omega}) \times \mathbf{p}.$$

Putting this into Eqs. (13.4) and (13.5), collecting terms for $\delta\mathbf{a}, \delta\mathbf{u}, \delta\vec{\omega}, \delta\mathbf{p}$, etc. and setting the coefficients equal to zero, yields:

$$\begin{array}{ll} \mathbf{S} = \mathbf{A} + \frac{d\mathbf{P}}{\mathbf{dt}} & \mathbf{D} = \mathbf{C} + \frac{d\mathbf{Q}}{\mathbf{dt}} - \mathbf{p} \times \mathbf{Q} \\ \frac{\partial J}{\partial \mathbf{u}} = \mathbf{R} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{P} & \frac{\partial J}{\partial \mathbf{p}} = \mathbf{C} + \frac{d\mathbf{Q}_1}{dt} + \mathbf{Q} \\ \frac{\partial J}{\partial \mathbf{u}'} = \mathbf{R}_1 + \frac{d\mathbf{P}_1}{dt} + \mathbf{P}_1 & \frac{\partial J}{\partial \mathbf{p}'} = \mathbf{C} + \frac{d\mathbf{Q}_2}{dt} + \mathbf{Q}_1 \end{array} \tag{13.6}$$

By repeated differentiation one gets:

$$\begin{aligned} \mathbf{P} &= \frac{\partial J}{\partial \mathbf{u}} - \mathbf{R}_1 - \frac{d}{dt}\left(\frac{\partial J}{\partial \mathbf{u}'} - \mathbf{R}_2 \right) + \frac{d^2}{dt^2}\left(\frac{\partial J}{\partial \mathbf{u}''} - \mathbf{R}_2 \right) - \cdots \\ \mathbf{Q} &= \frac{\partial J}{\partial \mathbf{p}} - \mathbf{C}_1 - \frac{d}{dt}\left(\frac{\partial J}{\partial \mathbf{p}'} - \mathbf{C}_2 \right) + \frac{d^2}{dt^2}\left(\frac{\partial J}{\partial \mathbf{p}''} - \mathbf{C}_3 \right) - \cdots \end{aligned} \tag{13.7}$$

Imagining that these functions of velocity and their derivatives to be computed, then the first of Eqs. (13.6) given and would be:

$$\frac{d\mathbf{P}}{dt} + \mathbf{R} = \mathbf{S} \quad \frac{d\mathbf{Q}}{\mathbf{dt}} + \mathbf{C} + \mathbf{p} \times \mathbf{Q} = \mathbf{D} \ . \tag{13.8}$$

The vectors **P** and **Q** are generalized linear and rotational momentum.

Eqs. (13.7) and (13.8) correspond, as is easy to verify, with the restraints derived from the usual variational formulations.

14. The transformation to quasistationary motion

In so far as we seek to treat the retarded potentials as series expansions, we must restrict our attention to treating motions for which the expansions as a function of time converge for time spans of the order of that needed for light to traverse the electron. In view of this fact we shall hereafter specialize our considerations to motions for which only a few terms in a Taylor series provide a good approximation, i.e., those which can be denoted 'quasistationary.'

From Eq. (13.8) one can always calculate the forces **S** and **D**, to which the electron must be subjected in order that the third derivatives of the coordinates vanish or at least are very small. That, however, is not the interesting problem. Rather we would like to know which quasistationary motions, if any, an electron can execute when essentially free or subject to relatively small external forces. To attack this problem without excessive effort, a certain amount of handwaving is unavoidable. To begin, I have allowed arbitrarily large translation and rotation velocities (except that the translation velocity must remain under the speed of light) but considered accelerations , $\mathbf{u}'$ and $\mathbf{p}'$, for which only the the first term is finite, i.e., $\mathbf{u}'' = 0$ and $\mathbf{p}'' = 0$, or at least nearly so. It turns out that such motions are possible only with relatively large external forces. For the case of small external forces, the translation velocity may be large, but rotational velocity must be constrained; the velocity of the surface of the electron must remain low. All this leads to the following situation. Either, one neglects the third power of the rotational velocity, whereupon it is permissable to include only $\mathbf{p}'$, as in the beginning, and to neglect $\mathbf{p}''$ so that the original expansions are sufficient, which, however, results in a loss of the most interesting manifestations of electron motion. Or, on the other hand, one includes the third power, which implies including $\mathbf{p}''$ so the original expansions must be extended. In sum, therefore, we see that for free or small external forces, electrons subject to quasistationary motion have rotational velocities for which the various quantities whose linear contribution to the surface of the electron, which here shall be denoted with ε, are to taken as indicated in Table 1:

Quantity	Order	Quantity	Order
$\mathbf{u}' : \left(\frac{\mathbf{u}'R}{Y^2}\right)$	ε^3	$\mathbf{p} : \left(\frac{\mathbf{p}R}{Y}\right)$	ε
$\mathbf{u}'' : \left(\frac{\mathbf{u}''R^2}{Y^3}\right)$	ε^4	$\mathbf{p}' : \left(\frac{\mathbf{p}'R^2}{Y^2}\right)$	ε^2
$\mathbf{u}''' : \left(\frac{\mathbf{u}'''R^3}{Y^4}\right)$	ε^5	$\mathbf{p}'' : \left(\frac{\mathbf{p}''R^3}{Y^3}\right)$	ε^3

TABLE 1

etc.

Based on this preparatory research, we now suggest the following approach. In Eq. (13.8) we drop all quantities of order ε^4, as specified above. It will turn out that this

equation then will be satisfied by a motion in which the velocities and their derivatives in fact have the sought order of magnitude. From this it follows that this motion for a limited time interval must in fact closely approximate the motion of an electron, and which will do so better, the smaller the value of ε, the lower the rotational velocity of the electron.

After the precision to degree ε^3 is established for Eq. (13.7), considerable simplifications result. The first differentiation of al terms of degree ε^0 with $\mathbf{u},\mathbf{p},\mathbf{u}',\mathbf{p}'$ etc. after a time give quantities of at least order ε^2 , since each term contains factors of $\mathbf{u}'$ or $\mathbf{p}'$ or higher and each subsequent differentiation raises the order by at least a factor of ε. Thus, the sought precision for the momenta is given by

$$\begin{aligned}\mathbf{P} &= \tfrac{\partial J}{\partial \mathbf{u}} - \mathbf{R}_1 - \tfrac{d}{dt}\left(\tfrac{\partial J}{\partial \mathbf{u}'} - \mathbf{R}_2\right),\\ \mathbf{Q} &= \tfrac{\partial J}{\partial \mathbf{p}} - \mathbf{C}_1 - \tfrac{d}{dt}\left(\tfrac{\partial J}{\partial \mathbf{p}'} - \mathbf{C}_2\right).\end{aligned} \tag{14.1}$$

Further, it is easy to see that to specify J adequately, it is necessary only to retain terms linear in $\mathbf{u}$ and $\mathbf{p}$, and for the accelerations terms in $\mathbf{u}'$ and $\mathbf{p}'$. Below we shall in fact calculate J to this extent; first we examine the terms $\mathbf{R},\mathbf{C}$ resulting from the variation δW.

15. Proof that δW vanishes for quasistationary motion

Eq. (12.5) for δW is:

$$\delta W = \frac{1}{8\pi}\int ds^0 \left[\delta\Phi\frac{\partial\Phi}{\partial\sigma} - \Phi\frac{\partial\delta\Phi}{\partial\sigma} - \delta\mathbf{A}\cdot\frac{\partial\mathbf{A}}{\partial t} + \mathbf{A}\cdot\frac{\partial\delta\mathbf{A}}{\partial t}\right].$$

and per Eq. (13.1) is also:

$$\delta W = \delta\mathbf{a}\cdot\mathbf{R} + \delta\vec{\omega}\cdot\mathbf{C} + \delta\mathbf{u}\cdot\mathbf{R}_1 + \delta\mathbf{p}\cdot\mathbf{C}_1 + \delta\mathbf{u}'\cdot\mathbf{R}_2 + \delta\mathbf{p}'\cdot\mathbf{C}_2 + \cdots,$$

To begin, consider the surface s over which the integral δW is evaluated, and which, as described above, is a sphere of a particlar radius centered on the electron. In the argument thus far, nothing would be changed were we to consider, in stead of s, a parallel, advancing surface. In fact we wish to consider such a surface, to be denoted by $\bar{s}$, which enclosed the volume $\bar{V}$ and for which all associated quantities will likewise be given an over-bar.

Considering the integral of δV, we see that Green's theorem, as used in §3, implies:

$$\delta W + \frac{d\delta Y}{dt} = \delta\bar{W} + \frac{d\delta\bar{Y}}{dt}, \tag{15.1}$$

i.e., that $\delta W + d\delta Y/dt$ is independent of s.

As Eq. (13.5) this expression was expanded in terms of expressions like:

$$\frac{d\mathbf{P}}{\mathbf{dt}} + \mathbf{R};\quad \frac{d\mathbf{Q}_2}{\mathbf{dt}} + \mathbf{Q}_1 + \mathbf{C}_2,$$

etc.—that then entered into the equation of motion. Each factor separately is a series in $\mathbf{u}',\mathbf{p}'$ etc. under the condition, that the surface s and volume V, over which δW and δV are to be integrated, are within the convergence domain of the the retarded potentials. Let such an expansion be indicated by the equation:

$$\frac{d\mathbf{P}}{\mathbf{dt}} + \mathbf{R} = \alpha_0 + \sum\alpha_{1,i}u_i' + \sum\alpha_{2,i}p_i' + \sum\sum\alpha_{3.i,j}u_i'p_j' + \cdots, \tag{15.2}$$

where each expansion coefficient α is actually an integral over s or V.

Now, as the whole expression $\delta W + d\delta Y/dt$ is independent of the choice of s, the same is true for each individual factor of each variation and each term α in the expansion of these factors. This is true clearly for all surfaces s that are in the domain of convergence; but,

as $\mathbf{u}', \mathbf{p}'$ can be chosen arbitrarily small, the convergence domain will be then arbitrarily large, leading to the following assertion:

If in the equation of motion (and Eq. (13.6)) the terms $\mathbf{R}, \mathbf{C}, \mathbf{P}, \mathbf{Q}$ *etc. are series expansions in* $\mathbf{u}', \mathbf{p}'$ *etc., then the expansion coefficients are independent of the choice of s. In particular, we may let* $s \to \infty$.

Of course, letting $s \to \infty$ can be done only after the series are determined.

If we want to show on the basis of this development that terms from δW vanish, we must estimate the size of two quantities, from one of which the behavior of Φ and $\mathbf{A}$ at infinity depends, and the other the individual terms from the rotation ε depend.

As for the first quantity, note the following. Let $\hat{r}$ be the radial distance from the center of the electron, now, if $\hat{r} \to \infty$, then the terms l and k from §2 also diverge. Thus, all terms in the expansion Eq. (11.7) also diverge. The terms depending on rotation in Eq. (11.8) for $\mathbf{A}$ require somewhat more analysis. The integral

$$\int d\xi d\eta d\zeta \rho\eta \frac{1}{k},$$

appears to diverge in the zeroth order, as l and k depend on the order of $\hat{r}$. Closer examination of the quotient $1/\hat{r}$, however, and one sees that at infinity

$$\frac{1}{k} = \alpha + \frac{\beta}{\hat{r}},$$

where α depends only on the coordinates $\vec{\xi}^0$, and no longer from $\vec{\xi}$, and β remains finite. Putting this into the integral, make it vanish when evaluated over α because of symmetry, and shows that the whole integral at infinity is of the order or $1/\hat{r}$. Likewise, all terms concerning rotation in $\mathbf{A}$ at infinity are on order smaller in the first approximation at infinity of k and l than that of $\hat{r}$.

To estimate the order of the second quantity in terms of ε, first we must determine to which order of exactitude we must know the terms $\mathbf{R}, \mathbf{C}, \mathbf{R}_1, \mathbf{C}_1$, etc. in order to get the equation of motion to order ε^s. Consider, for instance, $\mathbf{R}_1$, which can be decomposed into a part, $T(\mathbf{R}_1)$ which is independent of rotation and therefore denoted the "translation" component, and a rotation component $F(\mathbf{R}_1)$. The first part (compare Eq. (table p. x)) after one differentiation with respect to time, of order ε^3, and the second of order ε^2. Each subsequent differentiation raises the order by at least a factor of ε. From the equations of motion and momentum series, Eqs. (13.7) and (13.8), that one evaluate only those terms as given on the chart below:

R-terms	**C**-terms	order $\leq \varepsilon^n$
$\mathbf{R}$	$\mathbf{C}$	ε^3
$R(\mathbf{R}_1)$	$R(\mathbf{C}_1)$	ε^2
$T(\mathbf{R}_1)$	$T(\mathbf{C}_1)$	ε^0
$R(\mathbf{R}_2)$	$R(\mathbf{C}_2)$	ε
$T(\mathbf{R}_2)$	$T(\mathbf{C}_2)$	not at all

It seems now advantageous to expunge the terms $\mathbf{R}$ and $\mathbf{C}$, which has the consequence that the variation $\delta\mathbf{a} = \delta\vec{\omega} = 0$ is inserted into δW. In consequence, the above restrictions can be reformulated as follows

u-terms	**p**-terms	order
$\delta\mathbf{u}$	$\delta\mathbf{p}$	ε
$\delta\mathbf{u}'$	$\delta\mathbf{p}'$	ε^2

where all translation terms of order ε^2 are appended, so that, all in all, knowledge of δW to order ε^2 is needed.

Now, both behavior of the expansion of Eqs. (11.7) and (11.8) of the retarded potentials at infinity and simultaneously their order with respect to ε must be checked. In the following, a symbolic form will be given for the order of the retained terms shall be used. All terms of order ε^4 shall be neglected, as well as terms derived from a previously consider term by a factor of $1/\hat{r}$ or ε. The purely 'translation' terms shall be handled separately and denoted with a factor of τ.([8]) It is obvious, that the expansion of Eqs. (11.7) and (11.8) up to terms of order ε^2 are complete. That is:

$$\Phi = \tau\left(\frac{1}{\hat{r}} + \varepsilon^2\right) \quad \mathbf{A} = \tau\left(\frac{1}{\hat{r}} + \varepsilon^2\right) + \frac{\varepsilon}{\hat{r}^2} + \frac{\varepsilon^2}{\hat{r}} + \varepsilon^2.$$

The terms from the variation of Φ and $\mathbf{A}$ in Eqs. (11.7) and (11.8) depending on $\delta\mathbf{u}, \delta\mathbf{p}, \delta\mathbf{u}'$ and $\delta\mathbf{p}'$ are easily found. In view of the determined order of $\delta\mathbf{u}, \delta\mathbf{p}$ etc., one gets

$$\delta\Phi = \tau\left(\frac{\varepsilon 1}{\hat{r}} + \varepsilon^2\right) \quad \delta\mathbf{A} = \tau\left(\frac{\varepsilon}{\hat{r}} + \varepsilon^2\right) + \frac{\varepsilon^2}{\hat{r}^2} + \frac{\varepsilon^3}{\hat{r}}.$$

Note that, for the character of purely translation terms, the only important feature is that they be free of $\mathbf{p}, \mathbf{p}', \mathbf{p}''$, while the variations $\delta\mathbf{p}, \delta\mathbf{p}'$ are irrelevant.

Further:

$$\begin{aligned} \Phi\delta\Phi &= \tau^2 \tfrac{\varepsilon}{\hat{r}}\left(\tfrac{1}{\hat{r}} + \varepsilon\right), \\ \mathbf{A}\delta\mathbf{A} &= \tau^2\varepsilon\left(\tfrac{1}{\hat{r}^2} + \tfrac{\varepsilon}{\hat{r}} + \varepsilon^2\right) + \tau\tfrac{\varepsilon^2}{\hat{r}^2}\left(\tfrac{1}{\hat{r}} + \varepsilon\right) + \tfrac{\varepsilon^3}{\hat{r}^4}. \end{aligned} \tag{15.3}$$

The integrations in δW arise solely because one or the other factor of the product, either Φ or $\delta\Phi$ say, is subjected to the differentiation $\partial/\partial\sigma$. That is, following Eqs. (12.3) and (12.4):

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial n} + \mathbf{u}\frac{\partial}{\partial t} = \sum \cos(nx_i)\frac{\partial}{\partial x_i^0} + \mathbf{u}\left(\frac{d}{dt} + \mathbf{u}\cdot\frac{\partial}{\partial \mathbf{x}^0}\right),$$

it involves either a differentiation with respect to the coordinates of the point of interest or with respect to time while holding the relative coordinates of this point fixed. In view of the expansions of Eqs. (11.7) and (11.8) of Φ and $\mathbf{A}$, one sees that the first differentiation raises the order by $1/\hat{r}$ and the second by at least an order of ε.

Note that differentiation by $\partial/\partial\sigma$ of the factors of the product $\Phi\delta\Phi$ leaves translation factors as again translation factors. Such factors are surely those having the factor τ^2, as they can be only the product of two translation factors. On can then, according to the stipulation in Eq. (15.3) above, set $\tau^2 = \varepsilon^2$ (but not $\tau = \varepsilon$) and thereby get:

$$\Phi\delta\Phi = \frac{\varepsilon^2}{\hat{r}^2}, \quad \mathbf{A}\delta\mathbf{A} = \frac{\varepsilon^2}{\hat{r}^3} + \frac{\varepsilon^3}{\hat{r}^2}.$$

In regard to the differentiation $\partial/\partial\sigma$, if one multiples by the factor $1/\hat{r}$ or ε all the terms, at least those not disregarded because of the factor ε^4, they get then at least the power of $\hat{r}^3$ in the denominator and therfore vanish in the integral as s is taken to infinity.

The above stipulations on the above excluded quantities on $\mathbf{R}$ and $\mathbf{C}$ to order ε^3 are rendered superfluous. First it is obvious that $\mathbf{C}$ to all orders vanishes in so far as a rotation $\delta\vec{\omega}$ of an electron about is center changes nothing and $\mathbf{C}\cdot\delta\vec{\omega}$ is to represent the variation

[8] The requirement, tranlation terms to have attributed to them an order of ε^2 does not apply to Φ and $\mathbf{A}$, but only to δW, and will covered here with the factor τ.

of rotation from δW. On the other hand, one has from the variation of Φ and $\mathbf{A}$ that arise from the displacement $\delta\mathbf{a}$ and in δW deliver the term $\mathbf{R}\cdot\delta\mathbf{a}$:

$$\delta\Phi = -\delta\mathbf{a}\cdot\nabla\Phi; \quad \delta\mathbf{A} = -\delta\mathbf{a}\cdot\nabla\mathbf{A}.$$

If one now fixes the order of these variations as above by considering the order of $\delta\mathbf{a}$ as being order 0, while in this case neglecting the superfluous distinction of translation and rotation[9], one gets:

$$\begin{aligned} \delta\Phi &= \tfrac{1}{\hat{r}^2} + \tfrac{\varepsilon^2}{\hat{r}}, \qquad \delta\mathbf{A} = \tfrac{1}{\hat{r}^2} + \tfrac{\varepsilon^3}{\hat{r}}, \\ \Phi\delta\Phi &= \tfrac{1}{\hat{r}^3} + \tfrac{\varepsilon^3}{\hat{r}^2}, \quad \mathbf{A}\delta\mathbf{A} = \tfrac{1}{\hat{r}^3} + \tfrac{\varepsilon^3}{\hat{r}^2}. \end{aligned}$$

Introducing the factors $1/\hat{r}$ or ε to account for the differentiations in the determination of the integrands of δW, shows that all terms of order ε^3 also vanish when the surface over which the integrations are executed is taken to infinity. This leads to the conclusion: *All terms from δW in the equation of motion vanish when $s \to \infty$.*

It might appear that we need to examine the expansion of δY as $s \to \infty$. However, the following considerations render this unnecessary. The expansion coefficients α comprise some combination of $\mathbf{R}, \mathbf{C}, \mathbf{P}, \mathbf{Q}$ etc., as in Eq. (15.2), in either a surface or volume integrals. If s and V are finite, then the integrals are finite. But when $s \to \infty$, as we have just shown, all the coefficients in the integral remain finite, in fact vanish. This implies that the volume integral must remain finite as $V \to \infty$. Whatever values this integral delivers are immaterial for the determination of the equation of motion.

Thus, we may let $s \to \infty$ for all relevant terms and so obtain

$$\mathbf{R} = \mathbf{C} = \mathbf{R}_1 = \mathbf{C}_1 = \mathbf{R}_2 = \mathbf{C}_2 = 0. \tag{15.4}$$

16. Calculation of the kinetic potential J

The quantity J, which we define, in analogy to 'potential' of conventional mechanics, as the 'kinetic potential' of an electron, has the value:

$$2J = \int dv^0 \rho(-\Phi + \mathbf{v}\cdot\mathbf{A}),$$

and can be calculated up to linear terms in the accelerations $\mathbf{u}'$ and $\mathbf{p}'$, that is by exploiting the expansions, Eqs. (11.7) and (11.8) of Φ and $\mathbf{A}$, while neglecting terms in $\mathbf{p}''$. We emphasis again, of concern here are the impact points $\vec{\xi}$ of Φ and $\mathbf{A}$ that are to be taken over the volume of the electron, where $\mathbf{v} = \mathbf{u} + \mathbf{p}\times\vec{\xi}$.

This calculation can be simplified substantially by taking the translation velocity to lay on the x-axis and acceleration to be in the $x-y$ plane, so that:

$$u_x = \mathbf{u}; \quad u_y = u_z = u_z' = 0.$$

[9]This distinction is superflous because $\mathbf{R}$ in the equation of motion is not differentiated with respec to time, and only such differentiations affect the 'translation' terms.

Eqs. (11.7) and (11.8) take the following form:

$$k=\sqrt{(\xi^0-\xi)^2+(1-u^2)[(\eta^0-\eta)^2+(\xi^0-\xi)^2]},\quad l=u(\xi^0-\xi),$$
$$\mathbf{\Phi}=\mathbf{\Phi}^0+\mathbf{\Phi}^1,\quad \mathbf{g}=\mathbf{g}^0+\mathbf{g}^1,$$
$$\mathbf{\Phi}^0=\int\frac{dV\rho}{k},$$
$$\mathbf{\Phi}^1=\int\frac{dV\rho}{k^2}\left([u'_x(\xi^0-\xi)+u'_y(\eta^0-\eta)\frac{l^2-k^2}{2(1-u^2)}+uu'_x\frac{l^3-3k^2l-2k^3}{2(1-u^2)^2}\right),$$
$$g^0_y=\int\frac{dV\rho\xi_y}{k},$$
$$g^1_y=\int\frac{dV\rho\xi_y}{k^2}\left([u'_x(\xi^0-\xi)+u'_y(\eta^0-\eta)\frac{l^2-k^2}{2(1-u^2)}+uu'_x\frac{l^3-3k^2l-2k^3}{2(1-u^2)^2}\right),$$
$$\mathbf{h}=\int dV\rho\frac{l}{k}\vec{\xi},$$
$$A=\mathbf{u}\Phi+\mathbf{p}\times\mathbf{g}-\frac{\mathbf{u}'}{1-u^2}\int dV\rho\left(1+\frac{l}{k}\right)-\frac{\mathbf{p}'\times\mathbf{h}}{1-u^2}.$$

Here, for the sake of brevity, the integral $\mathbf{h}$ is introduced, and in Φ and $\mathbf{g}$ the acceleration free parts, Φ^0 and $\mathbf{g}^0$ are separated.

All these integrals have simple symmetry properties. For example

$$h_y=\int dV\rho\frac{l}{k}\eta=u\int\frac{d\xi d\eta d\zeta\eta(\xi^0-\xi)}{\sqrt{(\xi^0-\xi)^2+(1-u^2)[(\eta^0-\eta)^2+(\zeta^0-\zeta)^2}},$$

is odd in ξ^0 and η^0, and even in ζ^0. These symmetry characteristics will be useful by the determination of J, where $\vec{\xi}^0$ traverses the electron's whole volume. The second integral vanishes for any function that is uneven in any of the variables $\vec{\xi}$. In fact, one gets just by symmetry for J

$$2J=(u^2-1)\int dV^0\rho\Phi^0-\frac{u'_yp_z}{1-u^2}\int dV^0\rho\int dV\rho\left(\xi^0\frac{l}{k}+u(\eta^{02}-\eta^0\right)\frac{l^2-k^2}{2k^2}$$
$$+p_x^2\int dV^0\rho(\eta^0g^0_y+\zeta^0g^0_z)+p_y^2\int dV^0\rho(\zeta^0g^0_z+\xi^0g^0_x)+p_z^2\int dV^0\rho(\xi^0g^0_x+\eta^0g^0_y).$$

The integral multiplied by u'_y accepts another simplification. That is, by exchanging $\vec{\xi}$ and $\vec{\xi}^0$:

$$\int dV^0dV\rho^2\eta^2\frac{l^2-k^2}{2k^2}=\int dVdV^0\rho^2\eta^{02}\frac{l^2-k^2}{2k^2},$$

it can be seen that the second part vanishes; and, the first part can be written:

$$\int dV^0\rho\int dV\rho\frac{\xi^0u(\xi^0-\xi)}{k}=u\int\vec{d}V^0\rho\Phi^0\xi^{02}-u\int dV^0\rho\xi^0g^0_x.$$

If one defines:

$$M=\int dV^0\rho\Phi^0;\quad \mathbf{K}=\int dV^0\rho\vec{\xi}^0\mathbf{g}^0;\quad M'=-K_x+\int dV^0\rho\Phi^0\xi^{02},$$

then the full expression for J becomes

$$J=\frac{u^2-1}{2}M+\frac{p_x^2}{2}(K_y+K_z)+\frac{p_y^2}{2}(K_x+K_z)+\frac{p_z^2}{2}(K_x+K_y)-\frac{u'_yp_zu}{2(1-u^2)}M'.$$

The task now is to evaluate the integrals M,M' and K and then the potentials Φ and $\mathbf{g}^0$. To begin, one has

$$\Phi^0=\int\frac{d\xi\partial\eta\partial\zeta}{\sqrt{(\xi^0-\xi)^2+(1-u^2)[(\eta^0-\eta)^2+(\zeta^0-\zeta)^2]}},$$

an integral over a sphere centered on the origin of the system $\vec{\xi}$. The usual method of evaluating this integral consists of stretching the x-axis by the factor $1/(1-u^2)$, so that Φ^0 becomes the usual potential of attraction for a homogeneous ellipsoid of rotation. In this way one obtains:

$$\Phi^0 = \pi\rho[R^2E - \xi^{02}D - (\eta^{02}+\zeta^{02})D],$$
$$E = \tfrac{1}{u}\log\tfrac{1+u}{1-u} = u^2D + 2 = D + 2D_1.$$

The evaluation of $\mathbf{g}^0$, on the other hand, also involves an ellipsoid of revolution, but one that is not homogeneous, but rather one for which the density is proportional to the distance from the center. The methods used to evaluate homogeneous ellipsoids also leads directly to the result:

$$g_i^0 = \pi\rho\xi_i^0[R^2A_i + \xi^{02}B_i + (\eta^{02}+\zeta^{02})C_1]$$

where

$$A_x = E - \tfrac{1-u^2}{u}\tfrac{\partial E}{\partial u},$$
$$3B_z = -3D + \tfrac{1-u^2}{u}\tfrac{\partial D}{\partial u},$$
$$C_x = -D_1 + \tfrac{1-u^2}{u}\tfrac{\partial D}{\partial u}$$
$$A_x + 2A_y = E, \quad A_y = A_z,$$
$$3B_x + 2B_y = -3D, \quad B_y = B_z,$$
$$C_x + 4C_y = -3D_1, \quad B_y = C_z.$$

To evaluate the integrals M, M' and K, only elementary integrals of $\vec{\xi}^0$over the sphere V^0 need be carried out. The results are as follows:

$$M = \tfrac{16}{15}\pi^2R^5\rho^2E,$$
$$K_x = \tfrac{1}{7}\tfrac{16}{15}\pi^2R^7\rho^2\tfrac{E-2}{u^2},$$
$$K_y = K_z = \tfrac{1}{7}\tfrac{16}{15}\pi^2R^7\rho^2\tfrac{1-(1-u^2)E/2}{u^2},$$
$$M' = \tfrac{3}{7}\tfrac{16}{15}\pi^2R^7\rho^2\tfrac{1-(1-u^2)E/2}{u^2}.$$

After putting these expressions into J and dividing through by the constant factor $16\pi R\rho^2/15$, we get:

$$J = \frac{u^2}{2}M + \frac{p_x^2}{2}C_x + \frac{p_y^2}{2}C_y + \frac{p_z^2}{2}C_z - p_xu_{y'}uM, \tag{16.1}$$

where

$$\begin{aligned} M &= \tfrac{16}{15}\pi^2R^5\rho^2\left(1-(1-u^2)E/2\right)/u^2,\\ C_x &= \tfrac{2}{7}\tfrac{16}{15}\pi^2R^7\rho^2\left(1-(1-u^2)E/2\right)/u^2,\\ C_y = C_z &= \tfrac{1}{7}\tfrac{16}{15}\pi^2R^7\rho^2\left(-1+(1+u^2)E/2\right)/u^2,\\ M &= \tfrac{3}{7}\tfrac{16}{15}\pi^2R^7\rho^2\tfrac{1-(1-u^2)E/2}{2u^2(1-u^2)}, \quad E = \tfrac{1}{u}\log\tfrac{1+u}{1-u}, \end{aligned} \tag{16.2}$$

and for which the series expansions are:

$$\begin{aligned} M &= \tfrac{64}{15}\pi^2R^5\rho^2\left(\tfrac{1}{1\cdot3}+\tfrac{u^2}{3\cdot5}+\tfrac{u^4}{5\cdot7}+\cdots\right),\\ C_x &= \tfrac{2}{7}\tfrac{64}{15}\pi^2R^7\rho^2\left(\tfrac{1}{1\cdot3}+\tfrac{u^2}{3\cdot5}+\tfrac{u^4}{5\cdot7}+\cdots\right),\\ C_y = C_z &= \tfrac{1}{7}\tfrac{64}{15}\pi^2R^7\rho^2\left(\tfrac{1}{1\cdot3}+\tfrac{u^2}{3\cdot5}+\tfrac{u^4}{5\cdot7}+\cdots\right),\\ M' &= \tfrac{3}{14}\tfrac{64}{15}\pi^2R^7\rho^2\left(\tfrac{1}{1\cdot3}+\tfrac{u^2}{3\cdot5}+\tfrac{u^4}{5\cdot7}+\cdots\right)\tfrac{1}{1-u^2}. \end{aligned} \tag{16.3}$$

Finally, the special orientation of the coordinate system along the velocity and acceleration is abandoned in favor of an arbitrary orientation; in which case J, as it would also by

a simple coordinate rotation, becomes:

$$J = \frac{\mathbf{u}^2}{2}M + \frac{\mathbf{p}^2}{2}C_y - \frac{(\mathbf{p}\cdot\mathbf{u})^2}{\mathbf{u}^2}\frac{C_y - C_z}{2} - M'(\mathbf{p}\cdot(\mathbf{u}'\times\mathbf{u}). \tag{16.4}$$

17. THE ELECTRON'S EQUATION OF MOTION

The preceeding results lead to the equation of motion of an electron. Eqs. (13.6, 13.7) and (13.8), in view of the vanishing of the the quantities **R** and **C**, the simplification of the momentum to the terms in Eq. (14.1) and the absence of $\mathbf{p}'$ in the expressions for J, take on the following form:

$$\begin{gathered}\mathbf{P} = \tfrac{\partial J}{\partial \mathbf{u}} - \tfrac{d}{dt}\tfrac{\partial J}{\partial \mathbf{u}};\quad \mathbf{Q} = \tfrac{\partial J}{\partial \mathbf{p}},\\ \tfrac{d\mathbf{P}}{dt} = \mathbf{S};\quad \tfrac{d\mathbf{Q}}{dt} = \mathbf{p}\times\mathbf{Q} + \mathbf{D}.\end{gathered} \tag{17.1}$$

These equation in various respects bear a similarity to the equations of motion for a solid body from ordinary mechanics; i.e.:

1.) By absence of an external displacement force **S**, the linear momentum **P** is fixed in space.

2.) Equations for the rotational momentum **Q** have the very same form as the general Euler equations, except that the signs of the first two terms are exchanged. This implies that by absence of torques, the rotational momentum, unlike the linear case which stays fixed in space, remains fixed within the electron. This inversion of the electron's orbital structure evades the situation, that the kinetic potential depends on a rotational velocity **p** about a fixed axis in space, rather on an axis moving with the electron, so that the ellipsoid of momentum depends on the translation direction and does not rotate with the body of the electron.

3.) Given external forces from a potential U, the equation of motion then follows from the minimalization principle

$$\delta\int_{t_0}^{t_1} dt(J-U) = 0.$$

In this case J depends not only on velocity, as in the usual formulation, but also on acceleration, so that be a variation both the coordinates and the velocities must be held constant at the boundaries of the time interval.

If rotation is about the direction of translation ($p_z = 0$ in Eq. (16.1) for J), or if the translation velocity is zero (and therefore $M' = 0$), one arrives at ABRAHAM's "extraordinary" motion in which accelerations are absent from the potential J, and the only distinction to ordinary mechanics is that J is no longer a quadratic function of velocity.

4.) As the value of the kinetic potential depends obviously only on the relative positions of the three vectors $\mathbf{u}, \mathbf{u}'$ and **p**, nothing is changed if these three vectors are subjected to the same rotation. Thus, the following equation holds:

$$\mathbf{u}\times\frac{\partial J}{\partial \mathbf{u}} + \mathbf{u}'\times\frac{\partial J}{\partial \mathbf{u}'} + \mathbf{p}\times\frac{\partial J}{\partial \mathbf{p}} = 0.$$

In turn, this equation leads to

$$\frac{\partial J}{\partial \mathbf{p}} + \mathbf{u}\times\frac{\partial J}{\partial \mathbf{u}'} + \mathbf{a}\times\mathbf{P} = \int dt(\mathbf{D} + \mathbf{a}\times\mathbf{S}). \tag{17.2}$$

These equations correspond the the surface variation of ordinary mechanics, whereas the center of mass variation corresponds to

$$\mathbf{P} = \frac{\partial J}{\partial \mathbf{u}} - \frac{d}{dt}\frac{\partial J}{\partial \mathbf{u}'} = \int dt\mathbf{S}. \tag{17.3}$$

The right sides of these equation agree up to the neglected high order terms to the components of ABRAHAM*'s "quantities of motion" for the electron.*

18. FORCE FREE ELECTRON MOTION

When the external forces **S** and **D** vanish, so that the electron's motion is constant, the surface- and center of mass variations yield the intermediate integrals of the equation of motion, namely:

$$\mathbf{P} = \frac{\partial J}{\partial \mathbf{u}} - \frac{d}{dt}\frac{\partial J}{\partial \mathbf{u}'} = \text{const.} = \vec{\beta}$$

$$\frac{\partial J}{\partial \mathbf{p}} + \mathbf{u} \times \frac{\partial J}{\partial \mathbf{u}'} + \mathbf{a} \times \vec{\beta} = \text{const.} = \vec{\gamma}.$$

The next question is: whether and under which conditions, it is possible to have uniform translations in combination with rotation about an axis fixed in space?

Letting the translation direction be along the x-axis, gives:

$$a_x = ut; \quad \mathbf{a} - a_x = \mathbf{u} - u_x = \mathbf{u}' = 0.$$

In the surface variation the terms $a_x\beta_y$ and $a_x\beta_z$ increase linearly in time, while all other terms remain constant. For uniform motion, it must be that

$$\beta_y = \beta_z = 0; \quad P_y = P_z = 0.$$

The direction of the linear force coincides with the direction of motion. Thus, the consequences of the surface variation is exhausted as the left sides are constant.

Regarding the center of mass variation,, in view of the orientation choice made here, by differentiation of J, one gets:

$$P_y = \tfrac{\partial J}{\partial u_y} = -p_x p_y \tfrac{C_y - C_x}{2u},$$
$$P_z = \tfrac{\partial J}{\partial u_z} = -p_x p_z \tfrac{C_z - C_x}{xu}.$$

If both are to vanish, either $p_x = 0$ or $p_y = p_z = 0$. Thus, we get:

Uniform translation accompanied by rotation of an electron is possible only if the axis of rotation is parallel or perpendicular to the direction of translation.

By variable translational motion with rotation, the extraction of exact integrals is infeasible as neglecting higher order terms is not physically justifiable. Therefore, we shall seek solutions confined to being near those for uniform motion, to the extent that the terms correspond to those in Table 1, namely:

Variable	Order
$\mathbf{p}$	ε
$\mathbf{p}'$	ε^2
$\mathbf{p}''$	ε^3
$\mathbf{u}'$	ε^3

"Being near" the solution for uniform motion means here, that both u_y and u_z are restrained to order ε^2 and only u_x is allowed to take arbitrary values.

This whole investigation from the start was limited to obtaining the equations of motion to an exactitude of the order ε^3. This level of precision implies that the surface and volume variations can have an order of ε^2, as differentiation raises the level by one order.

The partial derivatives of J where terms up to order ε^2 have been kept, are:

$$\begin{aligned}
\tfrac{\partial J}{\partial u_x} &= u_x\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right)+\tfrac{p_x^2}{2}\tfrac{\partial C_x}{\partial u}+\tfrac{p_y^2+p_z^2}{2}\tfrac{\partial C_y}{\partial u}, \quad \tfrac{\partial J}{\partial u}=0,\\
\tfrac{\partial J}{\partial u_y} &= u_y\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right)-\tfrac{C_y-C_x}{u}p_xp_z, \qquad\qquad \tfrac{\partial J}{\partial u}=-M'p_zu,\\
\tfrac{\partial J}{\partial u_z} &= u_z\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right)-\tfrac{C_y-C_x}{u}p_xp_z, \qquad\qquad \tfrac{\partial J}{\partial u}=+M'p_yu,\\
\tfrac{\partial J}{\partial p_x} &= p_xC_x, \quad \tfrac{\partial J}{\partial p_y}=p_yC_y, \quad \tfrac{\partial J}{\partial p_z}=p_zC_z.
\end{aligned}$$

From these expressions one gets the components of the linear momentum with the center of mass variation correct to order ε^2:

$$\begin{aligned}
P_x &= u_x\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right)-\tfrac{p_x^2}{2}\tfrac{\partial C}{\partial u}+\tfrac{p_y^2+p_z^2}{2}\tfrac{\partial C_y}{\partial u}=\text{const.}=\beta_x,\\
P_y &= u_y\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right)-\tfrac{C_y-C_x}{u}p_xp_y+uM'p_z'=\text{const.}=\beta_y,\\
P_z &= u_z\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right)-\tfrac{C_z-C_x}{u}p_xp_z+uM'p_y'=\text{const.}=\beta_z.
\end{aligned} \tag{18.1}$$

From the first of these expressions one can provisionally take it, that u_x and in fact u itself remain constant to second order.

For rotations, it is preferable to use the equations of motion themselves rather than the surface variation. To accuracy of degree ε^3, one gets:

$$\begin{aligned}
p_x'C_x &= 0,\\
p_y'C_y &= -p_xp_z(C_y-C_z),\\
p_z'C_z &= p_xp_y(C_y-C_x).
\end{aligned}$$

In so far as u varies at most to second order, one may take the C_i as (essentially the average of u) constant. Thus, the integrals of these equations give:

$$\begin{aligned}
&p_x=\lambda, \quad p_y=\mu\cos(nt=\chi), \quad p_z=\mu\sin(nt+\chi),\\
&n=\lambda(C_y-C_x)/C_y \quad (\text{where } \chi,\lambda,\mu \text{ are integration constnats.}
\end{aligned} \tag{18.2}$$

Putting the the resulting values for the rotational velocity in the center of mass variation then gives:

$$u_x\left(M+\frac{u}{2}\frac{\partial M}{\partial u}\right)=\beta_x-\frac{\lambda^2}{2}\frac{\partial C_x}{\partial u}-\frac{\mu^2}{2}\frac{\partial C_y}{\partial u},$$

which implies that $u_x=$ const. up to third order. In addition:

$$\begin{aligned}
u_y\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right) &= \beta_y+\cos(nt+\chi)n\mu\tfrac{C_y-M'u^2}{u},\\
u_z\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right) &= \beta_z+\sin(nt+\chi)n\mu\tfrac{C_y-M'u^2}{u}.
\end{aligned} \tag{18.3}$$

The coordinate system can always be chosen such that $\beta_y=\beta_z=0$; in which case integration of the coordinates of the center of mass of the electron, **a**, are:

$$\begin{aligned}
a_x &= u_xt,\\
a_y\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right) &= \mu\sin(nt+\chi)\tfrac{C_y-M'u^2}{u},\\
a_z\left(M+\tfrac{u}{2}\tfrac{\partial M}{\partial u}\right) &= -\mu\cos(nt+\chi)\tfrac{C_y-M'u^2}{u}.
\end{aligned} \tag{18.4}$$

In words, the significance of Eqs. (18.2), (18.3) and (18.4) is: *A freely moving electron, i.e., one subject to no external forces, moves, not uniformly according to the conservation of inertia, rather its center of mass moves on a cylindrical spiral so that the electron precesses about the axis of the spiral such that the spiral axis, rotational axis and velocity of the center of mass remain in a plane.*

In principle it should be checked if the magnitude of all these quantities, taking the rotational velocity to first order as the standard, correspond to initial assumptions. This is easily done from Eqs. (18.2) and (18.3), and it turns out that in fact we have obtained an approximation to the motion of the electron by taking the the velocity of the surface of the electron to be small in comparison to the speed of light. Just how coarse this approximation is, is not to be estimated without great difficulty, but should be acceptable for velocities up to $c/10$ or even $c/5$.

The size of the electron's spiral can be estimated by considering the angular velocity of the fine motion, which is identical to the angular velocity about the spiral axis:

$$n = \lambda \frac{C_x - C_z}{C_y} = \lambda u^2 \vartheta,$$

$$\vartheta = \frac{1}{u^2} \frac{\frac{3-u^2}{2u} \log \frac{1+u}{1-u} - 3}{\frac{1+u^2}{2u} \log \frac{1+u}{1-u} - 1} = \frac{\frac{1}{3\cdot 5} + \frac{2U^2}{5\cdot 7} + \frac{3u^4}{7\cdot 9} + \cdots}{\frac{1}{1\cdot 3} + \frac{2u^2}{3\cdot 5} + \frac{3u^4}{5\cdot 7} + \cdots}$$

The factor ϑ goes from $1/5$ to 1 as u goes from 0 to 1. The inclination of the spiral is given by:

$$l = \frac{2\pi u}{n} = \frac{2\pi}{\lambda} \frac{1}{u} \vartheta.$$

The radius of the cylinder on which the spiral lays, is:

$$\rho = \frac{\mu R^2}{u} \vartheta',$$

where

$$\vartheta' = \frac{1}{R^2} \frac{C_y - M' u^2}{M + \frac{u}{2} \frac{\partial M}{\partial u}} = \frac{1}{7} \frac{\frac{5u^2+2}{2u^2} \log \frac{1+u}{1-u} - \frac{2}{u^2} - \frac{3}{1-u^2}}{\frac{1+u^2}{2u^2} \log \frac{1+u}{1-u} - \frac{1}{u^2}}.$$

The factor $\vartheta' = 1/7$ for $u = 0$, vanishes for $u = 0.77c$ and $\to \infty$ as $u \to 1$. For $u = 0.97c$ it equals 1.5.

The inclination of the spiral becomes:

$$\frac{2\pi\rho}{l} = R^2 \lambda \mu \frac{\vartheta'}{\vartheta}.$$

This implies that the spiral is very flat or stretched out whenever the rotational velocity on the surface of the electron, $R\lambda$ and $R\mu$, are small with respect to the speed of light, which is the case for all conditions considered herein.

If we apply this result for the spiral radius to cathode and BECQUEREL-rays for which $0.1c < u < 0.97c$, and therefore the rotational velocity μR is roughly the speed of light, *still the spiral radius is nearly equal to the electron radius. Only electrons for which the translation velocity is very small relative to the surface velocity μR, can the total motion take on visible dimensions.*

19. ELECTRON MOTION IN ELECTROMAGNETIC FIELDS. CONCLUSIONS

Consider a non rotating electron subject to translational forces **S**. In this case for the linear momentum, one has

$$\mathbf{P} = \mathbf{u} \left(M + \frac{u}{2} \frac{\partial M}{\partial u} \right) = \frac{\mathbf{u}}{u} \frac{\partial}{\partial u} \left(\frac{u^2 M}{2} \right),$$

from which it follows, that

$$\mathbf{S} = \frac{d\mathbf{P}}{dt} = \frac{\mathbf{u}'}{u} \frac{\partial}{\partial u} \left(\frac{u^2 M}{2} \right) + \frac{\mathbf{u}}{u} \frac{\partial}{\partial u} \left(\frac{1}{u} \frac{\partial}{\partial u} \left(\frac{u^2 M}{2} \right) \right) \mathbf{u} \cdot \mathbf{u}',$$

and if the x-axis is in the momentary direction of motion ($u_y = u_z = 0$):

$$\begin{aligned} u'_x \frac{\partial^2}{\partial u^2}\left(\frac{u^2 M}{2}\right) &= S_x, \\ u'_y \frac{1}{u}\frac{\partial}{\partial u}\left(\frac{u^2 M}{2}\right) &= S_y. \end{aligned} \tag{19.1}$$

The factors on u'_x and u'_y are in fact ABRAHAM's "longitudinal" and "transversal" mass. The second of these equations is what has been demonstrated by KAUFMANN's experiments.

When a rotating electron moves within the envelope of cathode- or BECQUEREL-rays as used in KAUFMANN's experiments, they experience both a lateral force and a torque. Simple estimates on the basis of the equations of motion show, however, that under these conditions *electrons execute an ultramicroscopic spiral motion, as described above, about a rotation axis which essential coincides with that from Eq. (19.1).*

The rotational velocity can increase up to the limits of this analysis, that is, up to approximately $c/5$. Thus, the final conclusion:

The results of the LORENTZ-WIECHERT *electrodynamics of a massless, rotating electron, as paradoxical as it may seem at first glance, do not conflict with any empirical evidence.*

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