

ON THE IMAGINABLE CONTENT OF QUANTUM THEORETICAL KINEMATICS AND MECHANICS

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A translation of:

**Über den anschaulichen Inhalt
der quantentheoretischen Kinematik und Mechanik**

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ABSTRACT. Herein exact definitions of the terms: location, velocity, energy, etc. (of electrons, for example), which have meaning in Quantum Mechanics also, will be examined. It will be shown that canonically conjugate quantities can be specified simultaneously only with a characteristic uncertainty (§1). This uncertainty is the actual reason for the appearance of statistics in Quantum Mechanics. Its mathematical formulation is captured by the DIRAC-JORDAN formalism (§2). From the basis of this formalism it is shown how macroscopic processes emerge from Quantum Mechanics (§3). For elucidation of the theory, gedanken experiments will be discussed.

INTRODUCTION

One considers that a physics theory can be intuitively understood if in simple situations it can be used to understand experiments and if one simultaneously finds that the theory applied to such experiments introduces no internal contradictions. For example, we believe, that we understand EINSTEIN's conception of a closed three dimensional space intuitively, because for us the experimental consequences of this notion are free of contradictions. Of course, this contradicts our usual notions of imaginable space-time. We can convince ourselves easily, however, that our usual space-time ideas for very large spaces comes from neither the laws of thought nor from experience. The imaginable meaning of Quantum Mechanics is still full of inner conflicts that influence the debate over discontinuous verses continuous theories, as well as particles verses waves. Already from these considerations one can conclude, that an interpretation for Quantum Mechanics with the usual kinematic and mechanical conceptions is not possible. Quantum Mechanics arose out of the attempt to go beyond the usual kinematic notions and to replace them with new relations between concrete experiences and numbers. Since this seems to have succeeded, the mathematical schemata of Quantum Mechanics does not need modification. Likewise a revision of geometry for very small spaces and short times, is not needed, as considering sufficiently weighty masses bring the the quantum mechanical laws arbitrarily close to classical laws, even when small space and short times are involved. But, it still seems that a revision of the kinematic and mechanical conceptions arises directly from the equations of Quantum Mechanics. Given a particular mass m , it, in our usual conceptions, has associated with it simply understandable qualities of 'location,' 'velocity' and 'center of mass,' etc. On the other hand, between mass, location and velocity in Quantum Mechanics, the relationship $\mathbf{p}\mathbf{q} - \mathbf{q}\mathbf{p} = -i\hbar$ obtains. Thus, here we have good reason to be suspicious of uncritical use of the terms: "location" and "velocity." If one acknowledges, that for processes in

very small spaces and short times, discontinuities are somehow typical, then the failure of these terms is plausible. If one considers the one dimensional motion of a mass point, then in continuum theory one can specify a time curve, an orbit for the particle, $x(t)$ (more precisely, its center of gravity) (See Fig. 1), for which the tangent is the instantaneous velocity. In discontinuous theory, on the other hand, instead of a curve, one has as sequence of points of finite separation. (See: Fig 2) In this case it is obviously meaningless to talk of a particular location as the velocity is defined by two points, and, likewise, two points determine a velocity.

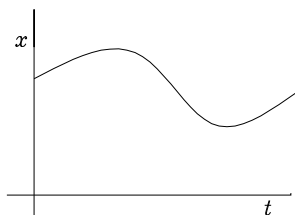


Fig. 1

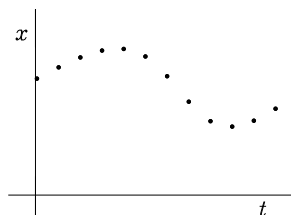


Fig. 2

Thus the question arises, whether through more exact analysis of its kinematic and mechanical notions, it would be possible to clarify the current conflicts in Quantum Mechanics and develop an intuitive understanding of quantum relationships.¹

1. THE CONCEPTIONS: PLACE, ORBIT, VELOCITY AND ENERGY

In order to be able to follow the quantum mechanical behavior of an object, one must know the fields that intermediate interaction with other objects. Only then can the Hamiltonian for the object be specified. [The following considerations are limited to nonrelativistic Quantum Mechanics, as the laws for quantum electrodynamics are still undeveloped.²] Further statements about the essential form (Gestalt) of an object are pointless; the best meaning to give the word "Gestalt," is the totality of these interactions.

If one seeks to be clear on just what the word "location" of an object, an electron say, is to signify (relative to a given coordinate system), then one must specify exactly which experiments are to be used to determine this location; otherwise, the word is empty. There is no deficit of experiments to call on, even for arbitrary precision; one might look with a microscope, for example. The maximum precision in this case is determined by the wave length of the light employed. Thus, in principle, one would chose a Γ -ray microscope so as to obtain optimal results. There is, however, a complication: COMPTON scattering. Each observation of an electron with scattered light presumes the use of a detector (eye, photographic plate, photo cell, etc.), and can be understood as photons encountering electrons so as to be scattered or reflected and then again redirected by the microscope lenses to finally cause the photoelectron effect. At the instant of the position measurement, i.e., when its trajectory is modified by the electron, the latter's trajectory is discontinuously altered. This alteration is bigger when the wave length of the scattered photons is smaller,

¹This paper arose out of efforts by others, originating long ago before the conception of Quantum Mechanics. I recall especially BOHR's work on the fundamental postulate of Quantum Mechanics, e.g., [1], and EINSTEIN's discussion about the relations between wave fields and light quanta. The clearest recent discussion of these problems is that by PAULI, [2], whom I heartily thank for fruitful discussions contributing to this paper.

²In recent times, however, DIRAC has made notable advances, e.g., [3].

that is the greater the precision of the measurement. Thus, the more precisely the position of the electron is determined, the greater the discontinuous modification of its momentum. So, again, we see a direct intuitive clarification of the relationship $\mathbf{pq} - \mathbf{qp} = -i\hbar$. Let q_1 denote the tolerance with which q is known (i.e., q_1 is the mean error in q), in this case the wave length of the measuring light, and p_1 the mean value of p , in this case the discontinuous alteration of the electron's momentum due to the COMPTON effect, then the elementary formula for the COMPTON effect is

$$(1.1) \quad p_1 q_1 \sim h.$$

Below we shall show that this relation is directly related with the commutation relationship $\mathbf{pq} - \mathbf{qp} = -i\hbar$. Here we note that Eq. (1.1) is a precise expression of the fact, that formerly was captured by the assertion that phase space is divided into cells of size h .

The specification of an electron's position can be determined also by other experiments, e.g., particle (electron) scattering. A precise measurement requires high speed particles, as slow particles, according to EINSTEIN, will have a long wave-length DE BROGLIE wave (the RAMSAUER effect) and thereby prevent precise measurements. Again, the more precise the determination of the location, the greater the discontinuous alteration of the target's moment as given also by Eq. (1.1).

This discussion seemingly has led to a clear understanding of the "location of an electron," and it remains only to consider the meaning of the term "size" of an electron. When two very fast electrons encounter each other in a very short interval Δt , then the two electrons were very close together at a separation Δl . According to the laws governing α -rays, we conclude, that Δl is of the order 10^{-12} cm. if Δt is sufficiently small and the particles sufficiently fast. All this makes sense, if we consider that the electron is a corpuscle with a radius smaller than 10^{-12} cm.

Let us now turn to the notion "electron orbit." By the term "orbit" we signify a sequence of points (with respect to a particular coordinate system) that an electron occupies serially. In so far as we already know what "location at a particular time" means, we encounter no new problems. Nevertheless it is easily seen, that the often used phrase: "the 1S orbit in hydrogen," from our viewpoint is meaningless. In order to measure this 1S "orbit," the atom must be illuminated with radiation for which the wave length is shorter than 10^{-8} cm. For such radiation, a single quantum is sufficient to eject the electron out of its orbit altogether (so that for such an orbit only one point in space can be determined); therefore, the term "orbit" here is meaningless. This can be seen, even without knowing the new theory, purely from experimental realities.

On the other hand, the imagined position measurement on the 1S orbit can be made. (Atoms in a given "stationary" state can be isolated with the STERN-GERLACH technique.) For a particular state, e.g., the 1S atomic state, there must be a probability function for the location of the electron, that corresponds to mean value over all phases and which is arbitrarily precisely determinable by measurement. According to BORN³ this probability function is given by $\psi_{1S}(q)\bar{\psi}_{1S}(q)$, if $\psi_{1S}(q)$ is the SCHRÖDINGER wave function for the 1S state. Together with DIRAC and BORN, and with an eye to latter generalizations, I wish to say: The probability is given by $S(1S, q)\bar{S}(1S, q)$, where $S(1S, q)$ is that column of the transformation matrix $S(E, q)$ from states with E to q , where $E = E_{1S}$ (E = energy).

³The statistical interpretation of DE BROGLIE waves was first suggested by EINSTEIN [4]. This conception played a large role in [5, Cap. 5, §3], and in [6]; thereafter it was mathematically analyzed and used to interpret scattering by BORN [7]. The foundation of the probability proposal in terms of transition theory of matrices can be found in [8–12]. A general discussion of the probability interpretation of Quantum Mechanics can be found in JORDAN [13] and BORN [14].

In that, for a particular quantum state, e.g., the $1S$ state, only the probability function for the electron's position can be given, one can see with BORN and JORDAN a characteristic statistical property of Quantum Mechanics in contrast to classical theory. On the other hand, if one chooses, say with DIRAC, that statistics arise through experimentation, then, obviously, even in the classical theory also, the probability of a particular electron position could be specified only so long as the phase of the atom is left unknown. But the difference between classical and quantum theory consists more in the fact, that in classical situations we can imagine determining the phase by a preparatory experiment. In fact, however, this is not possible, because every experiment to determine phase also changes or destroys it. In a particular stationary "states" of the atom, phase is in principle indeterminate, which can be seen as a direct result of the equations:

$$\mathbf{E}t - \mathbf{tE} = -i\hbar, \quad \text{or} \quad \mathbf{J}\omega - \omega\mathbf{J} = -i\hbar.$$

where \mathbf{J} is an action variable and ω is the corresponding angle variable.

The term "velocity" for an object can be defined easily with respect to experiments, if the object is not subject to forces. One could, for example, shine red light on the object and then using the DOPPLER Effect, deduce the object's velocity. The accuracy obtained thereby will be greater, the longer the wave length of the light used, in that the alteration of the object's velocity as a result of the COMPTON Effect will be less. Further, the object's location specification will be correspondingly uncertain according to Eq. (1.1). If the velocity of an electron shall be measured at a particular instant, then one must take it, that the nuclear charge and forces suddenly vanish, so that the motion can transpire force-free, thereby permitting the said procedure. Again one can easily convince oneself, that the function $p(t)$ for a particular state of the atom, $1S$ say, cannot be defined. Nevertheless, there exists a probability function of p for this state, that, according to DIRAC and JORDAN, has the value $S(1S, p)\bar{S}(1s, p)$, where again $S(1S, p)$ denotes that column in the transformation matrix $S(E, p)$ from energy \mathbf{E} to momentum \mathbf{p} , that belongs to the case $E = E_{1s}$.

In conclusion, we focus on experiments to measure the value of energy or the action variable \mathbf{J} , as only with their help can discontinuities in the values of these variables be discussed. FRANCK-HERTZ scattering, in view of quantum laws, allows measuring the energy of electrons executing straight line motion. Such experiments in principle can be carried out to arbitrary precision, if one abandons the simultaneous determination of electron location, i.e., phase, (compare with the above regarding p), in accord with $\mathbf{E}t - \mathbf{tE} = -i\hbar$. The STERN-GERLACH effect enables determining the magnetic or mean electric moment of an atom, that is, the measurement of the action variable \mathbf{J} . Phases remain undetermined in principle. Just as it is meaningless to talk about the instantaneous frequency of a light wave, so it is also meaningless to talk of the instantaneous energy of an atom. When employing the STERN-GERLACH Effect, this corresponds to the fact, that the precision of an energy measurement is the worse, the shorter the time interval is in which the atom is under the influence of the diverting force.⁴ An upper limit for the diverting force is given by the fact that the potential energy of these forces can vary within the atomic beam only by increments substantially smaller than the differences between the stationary states—if the stationary states are to be specifiable at all. Let E_1 be the energy increment, that satisfies this condition (E_1 also gives the precision of the energy measurement), thus, E_1/d is the highest value of the diverting force, if d is the width of the atomic beam (measurable in terms of the diffraction screen in use). The angular diversion of the atomic beam is then

⁴Compare with PAULI in [2, p. 61].

$E_1 t_1 / (dp)$, where t_1 is the time interval during which the atoms are under the influence of the diverting force, p is the momentum of the atoms in the beam direction. This diversion must be at least as large as the diffraction of the beam through a slit in order for a measurement to be possible. The angular diversion by diffraction is about λ/d where λ is the DE BROGLIE wave length. i.e.,

$$(1.2) \quad \frac{\lambda}{d} \sim \frac{E_1 t_1}{dp}, \quad \text{or} \quad \lambda = \frac{h}{p}; \quad E_1 t_1 \sim h.$$

This equation corresponds to Eq. (1.1) and shows, that a precise energy determination can be achieved only at the cost of an imprecise time measurement.

2. DIRAC-JORDAN THEORY

The results of the preceding Section can be encapsulated and generalized as the following statement: *All concepts, used in classical physics to describe a mechanical system, can be precisely defined for application to processes in atomic systems.* The experiments, for which such definitions pertain, practice shows, are afflicted with an intrinsic indeterminacy if conjugate variables are to be measured. The degree of this indeterminacy is given (as applied to any arbitrary pair of conjugate variables) by Eq. (1.1). Here, it may be interesting to compare Quantum Theory with Special Relativity. In relativity theory the word “simultaneous” cannot be defined except in the context of experiments for which the speed of light is an essential factor. Were it to be, that another, “more exact” definition of simultaneity existed, e.g., with instantaneous signals, then Relativity Theory would be impossible. In so far as instantaneous signals do not exist, and the definition of simultaneity does contain the speed of light as an essential factor, the possibility is opened for the postulate of the constancy of the speed of light, such that it does not conflict with the sense of the words “location, velocity, and time.” Similarly, the concepts “electron location” and “velocity” in Quantum Mechanics harmonize. All experiments that can be used to define these concepts, inevitably contain, according to Eq. (1.1), the specified indeterminacy, even while the concepts p and q as such can be precisely defined. Were there experiments that simultaneously made more precise determinations for p and q possible than allowed by Eq. (1.1), then Quantum Mechanics would be rendered impossible. The imprecision, that is specified by Eq. (1.1), engenders the tolerance for the validity of the relationship, that the quantum commutation finds expression as:

$$\mathbf{pq} - \mathbf{qp} = -i\hbar;$$

and it makes this equation possible without affecting the physical sense of the quantities p and q .

For those physical phenomena for which a quantum theoretical formulations still is to be found, (e.g., electrodynamic quantities), Eq. (1.1) is a challenge, which may even be useful for finding their quantum formulation. For Quantum Mechanics Eq. (1.1) can be extracted from the DIRAC-JORDAN formulation with minor generalization. If for the specific value η for some parameter the position q of an electron the value q' can be determined with a precision q_1 , then we can express this fact with the probability amplitude $S(\eta, q)$, that differs from null only in a small area about q' of size q_1 . In particular, one can set

$$(2.1) \quad S(\eta, q) \propto e^{-\frac{(q-q')^2}{2q_1^2} - \frac{i}{\hbar} p'(q-q')}, \quad \text{so that} \quad S\bar{S} \propto e^{-\frac{(q-q')^2}{q_1^2}}.$$

Then for the probability amplitude belonging to p

$$(2.2) \quad S(\eta, p) = \int S(\eta, q)S(q, p)dq.$$

Following JORDAN, for $S(q, p)$ one uses:

$$(2.3) \quad S(q, p) = e^{iqp/\hbar}.$$

Then according to Eq. (2.2), p will be noticeably different from zero only when $(p - p')q_1/\hbar$ is not substantially larger than 1. In particular for the case when Eq. (2.1) holds

$$S(\eta, p) \propto \int e^{\frac{i(p-p')q}{\hbar} - \frac{(q-q')^2}{2q_1^2}} dq,$$

that is:

$$S(\eta, p) \propto e^{-\frac{(p-p')^2}{2p_1^2} + \frac{i(p-p')}{\hbar}}; \text{ so that } \overline{SS} \propto e^{-\frac{(p-p')^2}{p_1^2}},$$

where

$$(2.4) \quad p_1 q_1 = \hbar.$$

The assumption of Eq. (2.1) for $S(\eta, q)$ corresponds, therefore, to the experimental fact, that the measured value of p' for p , implies the value of q' will be measured for q —with the limit on precision given by Eq. (2.4).

Seen purely mathematically, it is characteristic for the DIRAC-JORDAN formulation of Quantum Mechanics, that relations among $\mathbf{p}, \mathbf{q}, \mathbf{E}$ etc., can be written such that any given quantum quantity appears as a diagonal matrix. This feature is evident if one regards matrices as tensors (e.g., moment of inertia tensor) in a multidimensional space among which there are mathematical relations. One always can choose the axes for expressing these mathematical relationships such that they parallel the main axis of one of the quantities. Consequently one can characterize the mathematical relationship between two tensors A and B by the transform from a system oriented on A to one oriented on B . This latter variant corresponds to SCHRÖDINGER's theory. One can take from DIRAC's formulation the q -numbers as the "invariants" under all coordinate system transformations. If we seek to get physical results from this mathematical scheme, we must associate numbers to the quantum theoretical quantities, that is to the matrices (or tensors in the multi dimensional space). This is to be understood to mean, that in the multidimensional space for a particular given arbitrary direction (fixed by the nature of the considered experiment) it is asked, what the values of the matrix (e.g., the values of the moment of inertia tensor) in this direction would be. This question makes sense if the given direction coincides with one of the main axes of the matrix; in this case there is an exact answer. But also, if the given direction deviates only slightly from a main axis, one can still speak of a relative tendency to a certain probability of error for the matrix. One can say: that a quantum theoretical quantity may be seen to have a "value" to which a certain tolerance is attached; the tolerance depends on the coordinate system; for each quantum quantity there exists a coordinate system in which the tolerance for this quantity vanishes. A particular experiment can never yield precise values for all quantum quantities, rather it splits the quantities into "known" and "unknown" (better: more or less precise quantities) in a way characteristic of the experiment. The results of two experiments can be precisely compared only when they allow the identical division of quantum quantities into "known" and "unknown" categories (that is, when the tensors in each of their multidimensional spaces for both experiments can be viewed from the same axis). If the two experiments require different divisions into "known" and "unknown" then the experimental results can be related only in a statistical sense.

A precise discussion of such a statistical connection might be elucidated by a gedanken experiment. Consider a STERN-GERLACH atomic beam sent through a field F_1 , that is so strongly inhomogeneous, that noticeably many transitions are evoked through agitation. Thereafter the beam is allowed to propagate further unperturbed away from F_1 before encountering a similar second field F_2 . Between F_1 and F_2 , and after F_2 , let us suppose that, it is possible to measure the various stationary states by means of a magnetic field. The radiation forces on the atoms are taken to be negligible. If we know, an atom had the energy E_n , before it encountered F_1 , we can express this experimental fact, in that we attribute to it a wave function in p -space with energy E_n with the indeterminate phase β_n

$$S(E_n, p) = \psi(E_n, p) e^{-\frac{iE_n(\alpha + \beta_n)}{\hbar}}.$$

After traversing the field F_1 , surely this function is converted into⁵

$$(2.5) \quad S(E_n, p) \xrightarrow{F_1} \sum_m c_{nm} \psi(E_m, p) e^{-\frac{iE_m(\alpha + \beta_m)}{\hbar}}.$$

Herein the β_m are somehow arbitrarily determined, so that the c_{nm} are uniquely determined by F_1 . The matrix c_{nm} transforms the energy from the ‘before’ to ‘after’ values by the action of F_1 . Were we to measure the stationary states after traversing F_1 , using an inhomogeneous magnetic field for example, we would find with the probability $c_{nm}\bar{c}_{nm}$ that an atom changed from state n to state m . When we have determined that the final state is in fact m , then for subsequent calculations we should not use the function $\sum_m c_{nm} S_m$, rather S_m with undetermined phase. By fixing on state m we select from the totality of possibilities (c_{nm}) and simultaneously destroy (as will be explained below) all phase information that is in c_{nm} . The same process repeats itself by traversing F_2 ; in this case the transformation matrix is d_{nm} . If no measurement of the field is made between F_1 and F_2 , then the eigenfunctions are related as follows:

$$(2.6) \quad S(E_n, p) \xrightarrow{F_1} \sum_m c_{nm} S(E_m, p) \xrightarrow{F_2} \sum_m \sum_l c_{nm} d_{ml} S(E_l, p).$$

Let us set $\sum_m c_{nm} d_{ml} = e_{nl}$. If the stationary states after F_2 are measured, then with probability $e_{nl}\bar{e}_{nl}$ the final stationary state will be l . On the other hand if between F_1 and F_2 it is determined that in between them the state was m , then the probability of state l is given by $d_{ml}\bar{d}_{ml}$. By repeated repetitions of the same experiment, such that each time in between the two fields the state is determined, then after F_2 the state l will be observed with the relative frequency $Z_{nl} = \sum_m c_{nm}\bar{c}_{nm}d_{ml}\bar{d}_{ml}$. This expression does not agree with $e_{nl}\bar{e}_{nl}$. JORDAN[15] has spoken, therefore, about an ‘‘interference of probabilities.’’ The two experiments then, which lead to $e_{nl}\bar{e}_{nl}$ or Z_{nl} , are actually physically different. In one case the atoms suffer no disturbance between F_1 and F_2 , in the other case they are disturbed by the means of measuring the stationary state. This setup has the effect, that the ‘‘phase’’ of the atoms is in principle made indeterminate, just as a location measurement of an electron renders the momentum indeterminate (See: §1.). The magnetic field for determining the state between F_1 and F_2 disturbs the eigenvalue E ; observation of the orbit of the atomic beam (perhaps as a WILSON photograph) the atoms statistically and uncontrollably decelerate, etc. All this leads to the fact, that the final transformation matrix, e_{nl} (from the energy values preceeding F_1 to those after F_2), are no longer given by $\sum_m c_{nm}d_{ml}$, rather each summand has an undetermined phase. We can only expect, that the average of $e_{nl}\bar{e}_{nl}$ over all the phase variations equals Z_{nl} . A simple calculation verifies this surmise. — We can deduce, using certain statistical techniques, from one experiment the expected

⁵Compare with DIRAC [15] and BORN [16].

results of another. The first experiment selects from the possibilities a particular outcome, and thereby limits possibilities for all subsequent experiments. Such an interpretation for the transformation matrix S , or SCHRÖDINGER's equation, is possible only because the sum of solutions is also a solution. In this we see the deep meaning of the linearity of SCHRÖDINGER's equation; therefore, this equation can be seen only as a wave equation in phase space, and that any attempt to replace it with a *nonlinear* relativistic equation (for multiple electrons) is hopeless.

3. THE PASSAGE FROM MICRO- TO MACRO-MECHANICS

By virtue of the analysis in the preceding section of the terms "location, velocity and energy" of an electron, the quantum theoretical kinematic and mechanical concepts should be sufficiently clear to provide an understanding of macroscopic processes from the standpoint of Quantum Mechanics. The transition from micro- to macro-mechanics has been discussed previously by SCHRÖDINGER[17]; but, I do not believe that his equation settles the matter for the following reasons: According to SCHRÖDINGER, in a highly excited state a sum of eigen oscillations for a not altogether large wave packet can yield, by a periodic alteration of its size, the periodic motion of a classical "electron." Speaking against this notion are the following considerations. If a wave packet with the presumed character were to exist, then an atom would emit radiation expandable in a FOURIER series for which the higher terms are a whole number of times the fundamental frequency. The actual emitted radiations from an atom, however, are not, according to Quantum Mechanics, whole multiples of the ground state frequency—with the exception of the harmonic oscillator. SCHRÖDINGER's argumentation, therefore, is limited to handling the harmonic oscillator; in all other cases a wave packet would eventually fill the whole of space surrounding an atom. The greater the excited state, the longer the effect takes; but eventually it occurs. This same argument can be used against all attempts to directly transition between quantum and classical mechanics for highly excited states. Thus, attempts earlier to evade this argument by calling on the natural band width of stationary states; are clearly wrong, as this argument is precluded already for hydrogen by the meagerness of radiation from excited states. Furthermore, the transition to classical mechanics must be comprehensible also without reference to electrodynamics. BOHR has drawn attention repeatedly to these difficulties for linking quantum and classical mechanics.[1] Here we have clarified them again so thoroughly, because they seem to have been forgotten.

I believe, that one can formulate the emergence of a classical "orbit" thusly: *The "orbit" emerges as a consequence of observation.* Consider an atom in the 1000th excited state. The orbits here are quite large, so that in the sense of §1, the use of relatively long wave length light would suffice to determine the electron's location. If this determination need not be so extremely precise, then the COMPTON resistance force will lead to putting the atom into, say, a state between 950 and 1050; simultaneously the electron's momentum can be determined with the precision given by Eq. (1.1) using the DOPPLER Effect. This experimental fact can be characterized as a wave packet—in q -space with a size determined by the measuring light's wave length, which is built up of eigenfunctions between numbers 950 and 1000, as well as a corresponding wave packet in p -space. If after an appropriate interval, a second measurement with the same precision is made, its result, according to what was given in §2, can be specified only statistically, in so far as probable locations, only those within the above wave packets can be considered. This would be no different in classical theory, in that, in classical theory the second determination would be uncertain by cause of the uncertainty of the first. In addition, classical orbits would diverge in a manner

similar to the spread of wave packets. Nevertheless, the statistical laws are different in the two theories. The second determination selects out of all possibilities a particular “ q ” and thereby limits subsequent possibilities. After further location measurements, the results can be calculated by attributing to the electron a “small” wave packet with wave-length λ (the wave-length of the light used to make the measurement). That is, each measurement reduces the wave packet to its original size, λ . The values of the variables \mathbf{p} and \mathbf{q} are known during the experiment with a given exactitude. That the values of \mathbf{p} and \mathbf{q} *within this limit of precision* directly follow the equations of motion can be seen from the quantum laws:

$$(3.1) \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}; \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}.$$

The orbit can be calculated, however, as noted above, only statistically from the initial conditions, which can be seen as a consequence of the imprecision of these initial conditions. The statistical laws are different for classical and quantum mechanics, which can, depending on circumstances, lead to large macroscopic differences too. Before taking up a particular example, I wish to show how the above said transition to classical theory is mathematically formulated for a simple mechanical system, a force free particle. The equations of motion for this case, are:

$$(3.2) \quad \mathbf{H} = \frac{1}{2m}\mathbf{p}^2; \quad \dot{\mathbf{q}} = \frac{1}{m}\mathbf{p}; \quad \dot{\mathbf{p}} = 0.$$

As here time can be considered a parameter (a c-number), whenever there are no external forces, the solution to these equations is:

$$(3.3) \quad \mathbf{q} = \frac{1}{m}\mathbf{p}_0 t + \mathbf{q}_0; \quad \mathbf{p} = \mathbf{p}_0,$$

where \mathbf{p}_0 and \mathbf{q}_0 are the initial momentum and location at $t = 0$. At time $t = 0$ (See: Eqs. (2.1) to (2.4).), the value measured for $q_0 = q'$ has the precision q_1 , and $p_0 = p'$ has the precision p_1 . In order to get the values of \mathbf{q} at time t , according to DIRAC and JORDAN, the transformation functions which take matrices in which \mathbf{q}_0 is diagonal to those in which \mathbf{q} is diagonal must be found. \mathbf{p}_0 can be replaced by the operator $(-i\hbar\partial/\partial q_0)$. According to Dirac[15, Eq. (11)], the sought transformation amplitude $S(q_0, q)$ is given by the differential equation

$$(3.4) \quad \left\{ -\frac{t\hbar}{m} \frac{\partial}{\partial q_0} + q_0 \right\} S(q_0, q) = qS(q_0, q),$$

$$-\frac{t\hbar}{m} \frac{\partial S}{\partial q_0} = (q_0 - q)S(q_0, q);$$

$$(3.5) \quad S(q_0, q) = \text{const.} e^{-\frac{m \int (q - q_0) dq_0}{\hbar}}.$$

Thus, S is dependant on q_0 , that is, when at time $t = 0$, q_0 is known exactly, then for an arbitrary time $t > 0$ all values of t are equally probable, in other words, the probability, that q falls into a finite interval, is zero. This is actually intuitively clear. If q_0 is known exactly, this leads to an infinite COMPTON reaction. The same would hold true for any mechanical system. If, however, q_0 at time $t = 0$ is known only with a precision q_1 , and p_0 with precision p_1 (Compare with Eq. (2.1).), then

$$S(\eta, q_0) = \text{const.} e^{-\frac{(q_0 - q')^2}{2q_1^2} + \frac{p'(q_0 - q)}{\hbar}},$$

and therefore the probability function for q is to be calculated using the formula:

$$S(\eta, q) = \int S(\eta, q_0) S(q_0, q) dq_0.$$

The result is:

$$(3.6) \quad S(\eta, q) = \text{const.} \int e^{-\frac{m}{i\hbar} \left[q_0 \left(q - \frac{t}{m} p' \right) - \frac{q_0^2}{2} \right] - \frac{(q' - q_0)^2}{2q_1^2}} dq_0.$$

With the simplification

$$(3.7) \quad \beta = \frac{t\hbar}{mq_1},$$

the exponent in Eq. (3.6) becomes:

$$-\frac{1}{2q_1^2} \left\{ q_0^2 \left(1 + \frac{i}{\beta} \right) - 2q_0 \left(q' + \frac{i}{\beta} \left(q - \frac{t}{m} p' \right) \right) + q'^2 \right\}.$$

The term with q'^2 can be absorbed in the constant (i.e., as factor independent of q), so that integration yields:

$$(3.8) \quad \begin{aligned} S(\eta, q) &= \text{const.} e^{\frac{1}{2q_1^2} \frac{\left[q' + \frac{i}{\beta} \left(q - \frac{t}{m} p' \right) \right]^2}{1 + \frac{i}{\beta}}}, \\ &= \text{const.} e^{-\frac{\left(q - \frac{t}{m} p' - i\beta q' \right)^2 \left(1 - \frac{i}{\beta} \right)}{2q_1^2 (1 + \beta^2)}}. \end{aligned}$$

From this one gets:

$$(3.9) \quad S(\eta, q) \overline{S(\eta, q)} = \text{const.} e^{-\frac{\left(q - \frac{t}{m} p' - q' \right)^2}{q_1^2 (1 + \beta^2)}}.$$

The electron at time t will be found at location $tp'/m + q'$ with precision $q_1 \sqrt{1 + \beta^2}$. The wave packet, or better said, the "probability packet" has been expanded by the factor $\sqrt{1 + \beta^2}$. According to Eq. (3.7), β is proportional to the time t , inversely proportional to the mass m —this is obviously plausible—and inversely proportional to q_1^2 . Excessive precision in q_0 results in greater imprecision in p_0 which in turn leads to imprecision in q . The parameter η , which was introduced above for reasons of formality, can be dropped from all formulas, as it does not enter into calculations.

As an example that the difference in the statistics between classical and quantum theoretical considerations can lead to large macroscopic differences, let us consider briefly the deflection of electrons on a grating. When the grating constant is roughly the same as the DE BROGLIE wave-length of the electrons, then deflection occurs in discrete spacial directions, similar to light reflecting from a grating. Macroscopic classical theory here yields something quite different. Nevertheless we see no contradiction in terms of the orbit of a single electron. We could, if we could steer an electron to a particular location on the grating, determine that the reflexion was non classical. But if we could be so precise at locating the electron, that we could specify just where it impacted the grating, the electron would have to have a velocity so high that its DE BROGLIE wave-length would be so short, that in this approximation the reflexion actually could be given in a classically specifiable direction, without violating quantum principles.

4. CONSIDERATION OF A GEDANKEN EXPERIMENT

Following the interpretation of quantum theory as presented herein, the exact point in time of a “quantum jump” should be measurable just as certainly as the energy of a stationary state. The precision with which such a jump can be specified, is given by Eq. (1.2) as $h/\Delta E$ ⁶, if ΔE is the energy difference associated with the ‘jump.’ We imagine the following experiment: an atom at time $t = 0$ in the first excited state, seeks to drop down to the ground state. In analogy to Eq. (2.5), the eigenfunction

$$(4.1) \quad S(t, p) = e^{-\alpha t} \psi(E_2, p) e^{-\frac{iE_2 t}{\hbar}} + \sqrt{1 - e^{-2\alpha t}} \psi(E_1, p) e^{-\frac{E_1 t}{\hbar}},$$

can be attributed to the excited state, where the decay is described by the factor $e^{-\alpha t}$ (in reality it may not be so simple). To measure the energy, the atom might be sent through an inhomogeneous magnetic field, such as a STERN-GERLACH apparatus, such that the atom remains in the inhomogeneous field for considerable distance. The acceleration could be measured by breaking the orbit of the atomic beam into sections where at the end of each the deflection is determined. Depending on the velocity of the beam, this segmentation corresponds to dividing it into small time intervals Δt . From §1 Eq. (1.2), the interval Δt corresponds to a precision in the energy of $h/\Delta t$. The probability to obtain a particular energy value can be obtained from $S(p, E)$, and in the interval from $n\Delta t$ to $(n + 1)\Delta t$ calculated using:

$$S(p, E) = \int_{n\Delta t \rightarrow (n+1)\Delta t}^{(N+1)\Delta t} S(p, t) e^{\frac{iEt}{\hbar}} dt.$$

When at time $(n + 1)\Delta t$ it is determined that the atom is in the first excited state, then for all later time the eigenfunction Eq. (4.1) is no longer appropriate, rather one that evolves from it in which t is replace by $t - (n + 1)\Delta t$. On the other hand, if the atom is in the ground state, then the eigenfunction:

$$\psi(E_1, p) e^{-\frac{iE_1 t}{\hbar}},$$

obtains. Thus, it will be seen that for a series of Δt intervals, the atom is in an excited state, after which it will be seen to be continuously in the ground state. In this way, the instant of transition or ‘jump’ can be determined; but, the Δt intervals cannot be made smaller than $h/\Delta E$. In sum, the precision of determining the transition is limited to this value. This is the sort of experiment we have in mind when we speak of discontinuous energy transitions in the old quantum theory of PLANCK, EINSTEIN and BOHR. In so far as such an experiment is feasible, agreement on its outcome must be possible.

In BOHR’s fundamental quantum postulate the energy of an atom, as well as the action variable J , have precedence over other variables, e.g., the location of an electron, in terms of having their values fixed. This priority, that energy over other quantum quantities enjoys, is derived from the circumstance, that for closed systems it is an integral of the equations of motion (for which for the energy matrix, one has $\mathbf{E} = \text{const.}$); for open systems, the energy is not so distinguished. In particular, there are experiments for which the phase of an atom ω can be given precisely, but for which then the energy is undetermined in principle, corresponding to the equation $\mathbf{J}\omega - \omega\mathbf{J} = -i\hbar$, or $J_1\omega_1 \sim h$. Resonance florescence could provide such an experiment. If an atom is irradiated with an eigen frequency $\nu_{12} = (E_2 - E_1)/h$, the atom will oscillate in phase with the incoming signal, so that it makes no sense to ask which state, E_1 or E_2 the atom is in. The phase relation between atom and incoming radiation can be determined from the phase relationship of many atoms

⁶See: [2, p. 12].

among themselves (WOOD's experiment). If one wishes to avoid experiments with radiation, the phase relationships can be determined in that exact location measurements of the electron can be made in the sense of §1 at various times relative to the phase of incoming radiation (on many atoms). To an individual atom one can assign the wave function:

$$(4.2) \quad S(q, t) = c_2 \Psi_2(E_2, q) e^{-\frac{i(E_2 t + \beta)}{\hbar}} + \sqrt{1 - c_2^2} \Psi_1(E_1, q) e^{-\frac{iE_1 t}{\hbar}};$$

here c_2 depends on the amplitude and on β from the phase of the incoming radiation. The probability of a particular location q , is then

$$(4.3) \quad S(q, t) \overline{S(q, t)} = c_2^2 \Psi_2 \bar{\Psi}_2 + (1 - c_2^2) \Psi_1 \bar{\Psi}_1 \\ + c_2 \sqrt{1 - c_2^2} \left(\Psi_2 \bar{\Psi}_1 e^{-\frac{i}{\hbar}[(E_2 - E_1)t + \beta]} + \bar{\Psi}_2 \Psi_1 e^{\frac{i}{\hbar}[(E_2 - E_1)t + \beta]} \right).$$

The periodic term in Eq. (4.3) is experimentally distinguishable from the non periodic term, as a location determination for various phases of the incoming light can be made.

Let us consider a gedanken experiment proposed by BOHR in which an atomic beam, before passage through a STERN-GERLACH magnetic field, is excited by resonance fluorescence using a light beam. Thereafter, the atomic beam is observed both before and after being directed through the inhomogeneous magnetic beam. Before the magnetic field, resonance fluorescence occurs, that is, the atoms of the beam, analogous to the usual dispersion theory, are considered to re-emit spherical waves in phase with the stimulating beam. This last assumption, however, is in conflict with quantum theoretical principles, namely, that only a few atoms would be lifted into the "excited state," so that the resonance radiation would be emitted by only these few intensively excited centers. Thus, one used to say, the light quantum idea may be used here only for momentum-energy balance, but "in reality" only atoms in the ground state emit weak coherent spherical waves. After the atoms have passed through the magnetic field, there can be no doubt, that the beam consists of two parts, one of which is in the excited state, the other in the ground state. If now the atoms in the ground state radiate, this would be a violation of the energy balance in that one takes it, that all absorbed energy wound up in atoms in the excited state. More likely, there can be no doubt, that following the magnetic field only the beam of excited atoms emits light—and coherent light at that—from a few intensively radiating atoms. As BOHR indicated, this gedanken experiment makes it clear just how careful one must be with the concept of a "stationary state." Note now, from the understanding of quantum theory promulgated herein, it is particularly easy to discuss this experiment. The phases of the atoms are determined in the exterior radiation field, that is, it is pointless to speak about "the" energy of an atom. Also after exiting the radiation field, one may not say that the atom is in a stationary state, in so far as one seeks to find the radiation's coherence characteristics. One can, however, propose experiments to test just what state the atom is in, but the results can only be given statistically. Just such an experiment is effected by atoms traversing an inhomogeneous magnetic field. Following the magnetic field, the energies of the atoms are determined, in other words, the phases are undetermined. Radiation from them is incoherent and emitted only by excited atoms. The magnetic field determined the energies and therefore perturbed the phases. BOHR's gedanken experiment provides a very good clarification of the fact, that the energy of the atoms is not a number but a matrix. The conservation law pertains to the energy matrix and therefore also for the value of the energy as accurately as it is measured. Quantitatively, the perturbation of the phases can be formulated so: Let Q be the coordinate of the atom's center of gravity, such that instead

of Eq. (4.2), the eigenfunction

$$(4.4) \quad S(Q, t)S(q, t) = S(Q, q, t),$$

is assigned, where $S(Q, t)$ is a function, which [just as $S(\eta, q)$ in Eq. (3.8)] differs from zero only in a small neighborhood of the point in Q -space, and propagates in the direction of the radiation with the velocity of the atom. The probability of a relative amplitude q for an arbitrary value Q is given by the integral of $S(Q, q, t)S(Q, q, t)$ over Q , that is by Eq. (4.3).

The eigenfunction Eq. (4.4) will be substantially modified in the magnetic field and by cause of transitions to and from the excited states by some atoms, and will become

$$(4.5) \quad S(Q, q, t) = c_2 S_2(Q, t) \psi_2(E_2, q) e^{\frac{i(E_2 t + \beta)}{\hbar}} + \sqrt{1 - c_2^2} S_1(Q, t) \psi_1(E_1, q) e^{\frac{iE_1 t}{\hbar}}.$$

$S_1(Q, t)$ and $S_2(Q, t)$ are functions in Q -space that differ from zero only in small neighborhoods about some point; moreover, while these points will be different, the product $S_1 S_2$ will be zero everywhere. The probability of the relative amplitude q and a particular values Q is therefore:

$$(4.6) \quad S(Q, q, t) \overline{S(Q, q, t)} = c_2 S_2 \bar{S}_2 \psi_2 \bar{\psi}_2 + (1 - c_2^2) S_1 \bar{S}_1 \psi_1 \bar{\psi}_1.$$

The periodic term in Eq. (4.3) has vanished and with it the possibility to measure the phase relationship. The result of the statistical position determination will always be the same, independent of whatever phase the stimulating light had. We can take it, that the experiment with radiation, for which the theory is not yet developed, would give the same results on atomic phase relationships.

Finally, let us study the connection with Eq. (1.2), $E_1 t_1 \sim h$ with a complex of problems, discussed by EHRENFEST[18, 19] and others in two important publications in terms of BOHR's⁷ correspondence principle. EHRENFEST and TOLMAN speak of "weak quantization," whenever a quantized periodic motion is interrupted by quantum jumps or other perturbations into time intervals, which are not long relative to the period of the system, can be considered. In this case, not only the exact quantum energy values arise, rather with a qualitatively low a priori probability also energy values that deviate slightly from the exact values. Within quantum theory this behavior can be understood as follows: As the energy levels, because of exterior perturbations or quantum jumps, actually are modified, then every energy measurement, in as far as it is to be unambiguous, should be made in the interval between two perturbations. In this way an upper limit for t_1 in the sense of §1 is achieved. The energy value E_0 for a quantum state can be measured only with the precision given by $E_1 \sim h/t_1$. The question, whether for the system such energy values as E_1 , which should deviated from E_0 , "really" take on corresponding smaller statistical weight, or rather that their determination by measurement is subject to uncertainty, is in principle meaningless. If t_1 is shorter than the period of the system, then it is pointless to talk about stationary states of discrete energy values.

EHRENFEST and BREIT draw attention in this matter to the following paradox: A rotor, which we imagine to be a gear, is equipped with a gadget that after f rotations reverses the sense of rotation. The gear is engaged in a toothed rail that is movable back and forth between two masses, which force the rail and gear to reverse itself after a given number of revolutions. The actual period of the system is long in comparison to the rotation time of the gear; the energy levels are correspondingly dense, and the larger T , the more dense they will be. Then, from the usual quantum considerations, stationary states all have equal

⁷This connection was brought to my attention by PAULI.

statistical weights, so that for sufficiently large T , all energy levels will have the same frequency of occurrence—in contrast to what would be expected for a rotor. From our viewpoint, this paradox is made even sharper. That is, in order to determine whether the system has exclusively, or especially often, the energy belonging to the pure rotator, or whether it has all possible energy values with equal probability (i.e., values, that correspond to the small steps h/T), the time t_1 , that is small with respect to T (but $\gg t$) is sufficient; that is, although the large period for such measurements actually do not enter, express themselves, apparently, in that all possible energy values can arise. We are of the opinion, that such experiments to determine the total energy of the system would in fact yield all possible energy values equally likely; the cause behind this is not the large period T , but the movable linear rail. Even when the system finds itself in a state which corresponds to the energy of the quantum rotor, it can by cause of external forces working on the rail, suffer modification so as to wind up with a value not corresponding to one for the quantum rotor.⁸ The coupled system, rotor and rail, exhibits periodicity distinct from that of the rotor alone. The resolution of the paradoxes is likely to be found in the following considerations: if the rotor energy alone is to be measured, first it must be decoupled from the rail. In classical analysis, this could be effected by considering sufficiently small mass for the rail, so that the total system energy is essential that of the rotor alone. In quantum theory, on the other hand, the interaction between rail and gear is at least of the same order as the energy differences for the rotor alone (Even for a low mass rail the elastic interaction has a relatively high ground state!); for complete mechanical separation, each part takes on its own quantum energy levels. In so far as the energy values of the rotor alone can be measured, the precise energy values are those determined by the quantum values for this experiment. Even for a diminishingly small mass of the rail, the energy values for the coupled system differ from those of the rotor alone; the energy of the coupled system take on all possible values (admitted by T -quantization) with equal probability.

CONCLUSION

Quantum theoretical kinematics and mechanics differs substantially from the classical versions. The domain of applicability for the latter can not be derived either from the laws of thought or from experiment; this is a direct consequence of the relationship $p_1 q_1 \sim h$. In so far as ‘momentum, location and energy’ etc. of an electron and precisely defined concepts, one must not imagine that Eq. (1.1) is only a qualitative statement. Furthermore, as the experimental consequences of the quantum theory can be qualitatively comprehended, Quantum Mechanics can no longer be regarded as abstract and devoid of imaginable content.⁹ Clearly one would like, moreover, to derive the quantitative laws of quantum theory directly from imaginable fundamental notions, i.e., from Eq. (1.1). In this vain, JORDAN has attempted, to interpret the equation

$$S(q, q'') = \int S(q, q') S(q', q'') dq'$$

⁸According to EHRENFEST and BREIT this can seldom or never happen because of forces acting on the rotor itself.

⁹SCHRÖDINGER has charged that Quantum Mechanics is a formal theory of shocking, even revolting abstraction and obscurity. Certainly the mathematical and visualizable value of his theory cannot be over-valued. In matters of principle regarding physical problems, however, in my opinion the common understanding of his theory deviates from the notions of both EINSTEIN and DE BROGLIE on the one hand, and BOHR's Quantum Mechanics on the other.

as a probability relationship. However, we cannot endorse this viewpoint (§2). Rather, we believe, that the quantitative laws can be understood in the first place only on the basis of the imaginable foundations according to the principle of greatest simplicity. If, for example, the X -coordinate of an electron is not a “number,” as can be experimentally seen, is in accord with Eq. (1.1), then the simplest assumption [not contradicting Eq. (1.1)] is, that the coordinate is a diagonal element of a matrix, for which the off diagonal elements express an inexactitude, or alternately the inexactitude of transformations to other forms (See, e.g., §4). The assertion that the velocity in the X -direction is “in reality” a matrix is no more abstract and unimaginable than the statement that an electric field “in reality” is the time component of an antisymmetric tensor in space-time. The notion “in reality” is just as much or as little justified here as it would for any other mathematical entity applied to natural objects events. As soon as one accepts, that all quantum theoretical quantities “in reality” are matrices, the quantitative quantum laws follow without difficulty.

If one accepts, that the interpretation presented herein is largely correct in its essential points, then one is allowed a few words regarding its consequences. The first of these is, that it is *not* our conception that Quantum Theory, in contrast to classical physics, is in principle a statistical theory, that can draw from precise data only statistical conclusions. The renowned experiments, for example, of GEIGER and BOTHE, speak against this conception. Rather, in all cases in classical physics in which there are relationships between exactly measurable quantities, the corresponding relationships in quantum theory (momentum and energy conservation, say) are also exact. But, with respect to the strictly formulated principle: ‘if we know the present precisely, then we can calculate the future precisely,’ it is not the conclusion, but the hypothesis that is false. It is the present state that in principle cannot be determined with limitless precision. Thus all perception is a selection from a surfeit of existing possibilities, which then force a restriction on future projections. In that the statistical character of quantum theory is so tightly bound to the imprecision of perception, one might be seduced to consider, that behind the statistical world there is a “real” world hidden in which determinism rules. Such speculations seem to us, let us explicitly emphasize, otiose and meaningless. Physics should describe formally only the relationships between perceptions. In fact, one can characterize better the substance of our viewpoint as follows: because all experiments are subject to quantum laws and thereby Eq. (1.1), the principle of determinism definitely has been rendered invalid.

Note added in proof: After completion of this paper, further results from BOHR have lead to a deepening and refinement of my analysis of the implications of quantum principles. In this regard, BOHR has brought to my attention several substantial points that I had overlooked. First among them, the difference in observations does not exclusively depend on the occurrence of discontinuities, rather arises directly from the attempt to handle various effects simultaneously, i.e., those that arise both on the one hand in particle theory, and on the other in wave theory. For example, the employ of the proposed Γ -ray microscope presumes that the divergence of the beam is taken into consideration; which leads immediately to the fact that the direction of the COMPTON scattered electron can be known only imprecisely, again implying Eq. (1.1). Moreover, I have not emphasized sufficiently, that the simple theory of the COMPTON effect is limited to unconstrained electrons. The consequence of taking this into consideration by use of the uncertainty relationship, BOHR stresses, has serious consequences for any discussion on the passage from micro- to macro-mechanics. Finally, my considerations based on resonance fluorescence are not quite correct, because the connection between the phase of light and the electron orbit is

not as simple as assumed herein. I am indebted deeply to Prof. BOHR for access to, and discussion of, his upcoming paper on the conceptual construction of Quantum Mechanics.

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