# CARTAN'S PRINCIPLE OF DYNAMICS

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*A translation of:*
**Le principle dynamic d'E. Cartan**
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ABSTRACT. Cartan's Principle of dynamics is presented in traditional notation. It is shown to be equivalent to Hamilton's formulation, and is applied then to 1) a free particle, 2) a system of particles with holonomic constraints, 3) a free particle in General Relativity and 4) a system subject to nonholonomic constrains.

## INTRODUCTION

We are indebted to Elie Cartan for a principle of dynamics equivalent to that of Hamilton which he called the 'principle of momentum-energy conservation.' [1, p. 7-14]. Cartan's Principle, although more abstract than Hamilton's, has certain advantages. It is based on a linear differential form, $\omega$ , for which the coefficients have direct mechanical significance; whereas Hamilton's Principle is based on minimizing the action, a function without direct mechanical significance. There is an additional important difference between the two principles. In Hamilton's theory, time plays a privileged role, while Cartan's Principle gives the laws of mechanics a form independent of the representation of space-time.

Cartan's Principle, as far as I know, is not well known by specialists interested in Mechanics. In view of this situation, we propose presenting Cartan's Principle here in a manner differing from that in his book. For this purpose we shall call on the Lie derivative, which was not known at the time Cartan wrote, but is more appropriate for explicating his principle.

To begin, we take up the study of two applications of Cartan's Principle that he treated also: 1) the case of point particle subjected to a force derived from a function of the coordinates at the particle's point and time, and 2) the general case of point particles subject to holonomic constraints dependant on time. Then we also consider the following cases: 3) that of a free particle subject to a gravitational field as given by General Relativity and finally 4) a system of particles subject to nonholonomic constraints.

The formulas of Lie differentiation shall not be recapitulated, an appendix with the pertinent details necessary to justify our calculations is attached.

## 1. A FREE POINT PARTICLE

Suppose that in a Euclidean space $E_3$, equipped with an orthogonal coordinate system, there is a free particle of mass $m$ subjected to a force derived from a function $U$ of the particle's coordinates in space, $x^\varkappa$ $(\varkappa = 1,2,3)$, and of its time, $t$. The equations of motion are of the following form:

$$m\ddot{x}^\varkappa = \frac{\partial U}{\partial x^\varkappa} \quad \left(\ddot{x}^\varkappa = \frac{d^2x^\varkappa}{dt^2}\right), \tag{1.1}$$

and its kinetic energy is given by the formula

$$T = \frac{1}{2} m\dot{x}^2, \quad x = (x^1, x^2, x^3). \tag{1.2}$$

Eqs. (1.1) form a system of second order differential equations; to transform them into first order equations, we set

$$p^{\varkappa} = m\dot{x}^{\varkappa}, \tag{1.3}$$

and we consider the affine space $E_7$ (state space) with coordinates $x^i, p^i, t$. The point $(x, p, t)$ of $E_7$ is called the "state" of the particle; and, the ensemble of states corresponding to a motion of the particle is denoted its *trajectory*. These trajectories, which shall appear frequently in the following, are devoid of concrete existence, even though the concept is very useful.

It is easily seen that the equations of a trajectory can be presented in the following form:

$$\frac{dx^{\varkappa}}{Q^{\varkappa}} = \frac{dp^{\varkappa}}{P^{\varkappa}} = \frac{dt}{1}, \tag{1.4}$$

where $Q^i, P^i$ are analytic functions[1] of the point $(x, p, t)$ in the space $E_7$. It is also easy to see that every motion of the particle corresponds to a trajectory and visa versa. Eqs. (1.4) correspond to an $E_7$-vector field:

$$\vec{V} = (Q^1, Q^2, Q^3, P^1, P^2, P^3, 1). \tag{1.5}$$

Let us consider now the one-form:

$$\omega = \vec{p} \cdot d\vec{x} - H dt, \tag{1.6}$$

where $H$ is defined by

$$H = T - U.$$

We note that all the coefficients of the form ω have a direct mechanical significance: the first three are the momentum and the last is the total energy of the point particle. Cartan called ω the 'momentum-energy tensor'; but, we shall give it a more convenient designation, namely, the 'elementary measure of momentum-energy.' It plays a fundamental role in Cartan's Principle.

To describe Cartan's Principle we imagine an arbitrary closed line in $E_7$ with the collection of trajectories issuing from it and forming a tube. Now, Cartan's Principle is stated so:

In order that Eq. (1.4) determine the trajectories of a free point particle, it suffices that the integral

$$\oint \omega,$$

evaluated on an arbitrary line making a closed circuit, a loop, around the tube, is invariant for Eqs. (1.4).

Likewise, this principle can be expressed also as follows [2, p. 401]:

$$L_V \omega = 0, \quad \underline{V} \rfloor \omega = 0. \tag{1.7}$$

(See the appendix for notation.) The second of these leads to the following relationship:

$$H = \sum_{\varkappa} p^{\varkappa} Q^{\varkappa}, \tag{1.8}$$

[1] Herein all functions are taken to be analytic.

and the first gives:

$$P^{\lambda} + \sum_{\varkappa} p^{\varkappa} \frac{\partial Q^{\varkappa}}{\partial x^{\lambda}} = 0, \tag{1.9}$$

$$\sum_{\varkappa} p^{\varkappa} \frac{\partial Q^{\varkappa}}{\partial x^{\lambda}} = 0, \tag{1.10}$$

$$\sum_{\varkappa} Q^{\varkappa} \frac{\partial H}{\partial x^{\varkappa}} + \sum_{\varkappa} P^{\varkappa} \frac{\partial H}{\partial p^{\varkappa}} - \sum_{\varkappa} p^{\varkappa} \frac{\partial Q^{\varkappa}}{\partial t} + \frac{\partial H}{\partial t} = 0, \tag{1.11}$$

where sums are over $\varkappa = 1,2,3$.

In view of Eq. (1.8), from Eq. (1.11) one deduces the equation:

$$\sum_{\varkappa} Q^{\varkappa} \frac{\partial H}{\partial x^{\varkappa}} + \sum_{\varkappa} P^{\varkappa} \frac{\partial H}{\partial p^{\varkappa}} = 0. \tag{1.12}$$

Likewise, on differentiating Eq. (1.8) with respect to $x^{\lambda}$, one finds:

$$\frac{\partial H}{\partial x^{\lambda}} = \sum_{\varkappa} p^{\varkappa} \frac{\partial Q^{\varkappa}}{\partial x^{\lambda}};$$

and substituting the right side into Eq. (1.9) gives:

$$P^{\lambda} = -\frac{\partial H}{\partial x^{\lambda}}. \tag{1.13}$$

If one then differentiates Eq. (1.8) with respect to $p^{\lambda}$, one gets:

$$Q^{\lambda} = \frac{\partial H}{\partial p^{\lambda}}. \tag{1.14}$$

We note here that Eq. (1.12) is also a consequence of Eqs. (1.13) and (1.14) .

On reexamining Eq. (1.4), and having computed Eqs. (1.13) and (1.14), one sees that it can be rewritten as:

$$\frac{dx^{\varkappa}}{\frac{\partial H}{\partial p^{\varkappa}}} = \frac{dp^{\varkappa}}{-\frac{\partial H}{\partial q^{\varkappa}}} = \frac{dt}{1}.$$

These equations conform to those of Hamilton; one sees that Cartan's Principle resolves completely the problem of determining the equations of motion for free point particle. We add also that Eqs (1.13) and (1.14) give the relation:

$$dH = \frac{\partial H}{\partial t} dt,$$

which expresses the 'live force' theorem (théorèm des forces vives).

## 2. A Particle system with holonomic constraints

While maintaining the notation of §1, let us suppose that in the space $E_3$ we have a system $S$ of $N$ point particles $s_h = m_h x_h$ $(h = 1,2,\ldots,N)$, where $m_h$ is the mass of $s_h$, and $x_h$ is a point in $E_3$.

Suppose further that the points $x_h$ are subject to restraints conforming to the equations:

$$x_h^{\varkappa} = x_h^{\varkappa}(q,t), \tag{2.1}$$

where $q = (q^1, q^2, \ldots, q^r)$ designate an analytic variety, $M_r$, finite and connected, of dimension $r$, and that $(q,t)$ is a point in the space $M_r \times t$. We see that the matrix[2]

$$\left[\frac{\partial x_h^{\varkappa}}{\partial q^{\rho}}\right]$$

s of rank $r$ and permits expressing the variables $q^{\rho}$ as functions of the variables $x_h^{\varkappa}$ and $t$:

$$q^{\rho} = \psi^{\rho}(x_h^{\varkappa}, t). \tag{2.2}$$

We suppose further that at each point $s_h$ there is an external force $\vec{X}_h = (X_h^{\varkappa})$, that can be derived from an analytic function $U(x_h^{\varkappa}, t)$ of the coordinates $x_h^{\varkappa}$ and the time $t$. These forces put the system of particles into motion according to the equations:

$$m_h \ddot{\vec{x}} = \nabla U, \tag{2.3}$$

and they provide an expression for the kinetic energy:

$$T = \sum_{h=1}^{N} \frac{1}{2} m_h \sum_{\varkappa} (\dot{x}_h^{\varkappa})^2. \tag{2.4}$$

We note now that Eq. (2.1) leads to the relationships:

$$\dot{x}_h^{\varkappa} = \frac{\partial x_h^{\varkappa}}{\partial q^{\rho}} \dot{q}^{\rho} + \frac{\partial x_h^{\varkappa}}{\partial t}. \tag{2.5}$$

We shall consider the quantities $q^{\rho}$ as independent variables and designate them: 'velocity components on the variety $M_r$.' Following from Eqs. (2.4) and (2.5), kinetic energy can be expressed as a function of the variables $(q^{\rho}, \dot{q}^{\rho}, t)$:

$$T = t(q, \dot{q}, t). \tag{2.6}$$

Another consequence of Eqs. (2.4) and (2.5) is that the second order terms of $\dot{q}$ in $T$ render it a second order positive definite form.

Let

$$p_{\rho} = \frac{\partial T}{\partial \dot{q}^{\rho}}; \tag{2.7}$$

the derivatives $\partial T / \partial \dot{q}^{\rho}$ are independent linear expressions with respect to the variables $\dot{q}^{\sigma}$. Following the method used in §1, we consider states, $(q, p, t)$ in the space $E_{2r+1}$ of system $S$. A motion of the system $S$, that is to say, a solution $x_h^{\varkappa} = f_h^{vk}(t)$ of the system of equations Eq. (2.3), corresponds to an ensemble of states forming a trajectory of the system $S$, and the equations for these trajectories can be presented in the form

$$\frac{dq^{\rho}}{Q^{\rho}} = \frac{dp_{\rho}}{P_{\rho}} = \frac{dt}{1}, \quad (\rho = 1, 2, \ldots, r), \tag{2.8}$$

where $Q^{\rho}$ and $P_{\rho}$ are analytic functions of the point $(q, p, t)$ in the space $E_{2r+1}$. Corresponding to the system of Eqs. (2.8), in the affine space $E_{2r+1}$ there exists the vector field:

$$\vec{V} = (Q^1, Q^2, \ldots, Q^r, P_1, P_2, \ldots, P_r, 1). \tag{2.9}$$

Consider now the form[3]

$$\omega = p_{\rho} dq^{\rho} - H dt, \tag{2.10}$$

[2]Greek indices in this Section take the values $1, 2, \ldots, r$.

[3]We use here, as is customary, the summation convention.

where $H$, determined by

$$H = T - U, \tag{2.11}$$

represents the total energy of the system $S$.

Applying reasoning analogous to that used in §1, consider a loop enclosing a bundle of trajectories constituting a tube. In so far as Eqs. (2.8) determine the motion of system $S$, it is necessary and sufficient, according to Cartan's Principle of dynamics, that the curvilinear integral equals zero, i.e.:

$$\oint_C \omega = 0,$$

if evaluated on a loop $C$ enclosing an invariant trajectory tube for Eqs. (2.8). This situation is expressed by the relations:

$$L_v\omega = 0, \quad \underline{\vec{V}|}\omega = 0. \tag{2.12}$$

Expanding these expressions one obtains Hamilton's equations of motion:

$$\frac{dq^\rho}{\frac{\partial H}{\partial p_\rho}} = \frac{dp_\rho}{-\frac{\partial H}{\partial q^\rho}} = \frac{dt}{1}, \quad (\rho = 1, 2, \ldots, r), \tag{2.13}$$

which demonstrates the equivalence of the principles of Cartan and Hamilton.

We add here that Cartan's Principle also leads to the equations of motion for solid bodies; in order to get them in the preceeding calculations, replace the summations over points of the system $S$ with integrations over the domain occupied by the solid body.

Eqs. (2.13), deduced from Eq. (2.10), give the laws of mechanics a form independent of the representation of $E_3$ space. Now we shall show, that one can formulate these equations such that they are independent of the representation of the space $E_3 \times t$, time does not play a privileged role [1, p. 14].

Recall that we expressed the kinetic energy $T$ and the potential energy $U$ of the system $S$ by means of $2r+1$ variables $q^\rho, \dot{q}^\rho, t$. Now, to achieve our goal, we shall replace the $r$ variables $\dot{q}^\rho$ by $r+1$ variables,( $\overline{q}^\rho, \overline{t}$), derived from the former by the relations:

$$\overline{q}^\rho = \overline{t}\dot{q}^\rho. \tag{2.14}$$

To present the quantities which refer to the system $S$ by means of the variables $(q^\rho, t, \overline{q}^\rho, \overline{t})$, we introduce new notation by positing:

$$\overline{p}_\rho = \overline{t}p_\rho, \quad \overline{H} = \overline{t}H, \quad \overline{\omega} = \overline{t}\omega.$$

Thus, one gets:

$$\overline{\omega} = \overline{p}_\rho dq^\rho - \overline{H}d\overline{t}. \tag{2.15}$$

To simplify the reasoning, we limit ourselves to the case, where the system $S$ is neutral, that is to say, where kinetic energy given by Eq. (2.6) contains only terms of second degree in $\dot{q}^\rho$. One can easily convince oneself that $\overline{H}$, homogeneous and of first degree in $(\overline{q}^\rho, \overline{t})$, is expressible by means of $q^\rho, t, \overline{q}^\rho, \overline{t}$.

In replacing the symbol $\omega$ by $\overline{\omega}$ in Eq. (2.12), one obtains, by virtue of Cartan's Principle, the equations:

$$\frac{dq^\rho}{Q^\rho} = \frac{d\overline{p}_\rho}{P_\rho} = \frac{d\overline{t}}{1},$$

where $\overline{Q}^\rho$ and $\overline{P}_\rho$ are functions of variables $q^\rho, t, \overline{q}^\rho, \overline{t}$.

These equations are covariantly related to the form Eq. (2.15) by the transformations of the variables $q^\rho, \overline{t}$ of space-time.

## 3. A FREE PARTICLE IN EINSTEIN SPACE

Let $q^\rho$ $(\rho = 1,2,3,4)$ be the coordinates of a point in the space $E_4$ where the metric is specified by the form

$$ds^2 = g_{\rho\sigma} dq^\rho dq^\sigma, \tag{3.1}$$

with signature (3.1); the variable '$s$' designates the proper time of a moving point particle (eigentime).

The kinetic energy of a material point with proper mass $m$ (eigenmass) subject to gravitational force, equals the expression:

$$T = \frac{1}{2} m g_{\rho\sigma} \dot{q}^\rho \dot{q}^\sigma, \quad \left(\dot{q}^\rho = \frac{dq^\rho}{ds}\right). \tag{3.2}$$

We define the components $p_\rho$ of momentum in the same manner as in §2 (See: Eq. (2.7):

$$p_\rho = \frac{\partial T}{\partial \dot{q}^\rho}, \tag{3.3}$$

which, in view of Eq. (3.2), leads to the equation:

$$p_\rho = m g_{\rho\sigma} \dot{q}^\rho. \tag{3.4}$$

Multiplying this equation by $g^{\rho\tau}$ and summing on $\rho$, yields:

$$p^\tau = g^{\rho\tau} p_\rho. \tag{3.5}$$

If one takes account of this formula, Eq. (3.2) becomes:

$$T = \frac{1}{2m} g^{\rho\sigma} p_\rho p_\sigma. \tag{3.6}$$

Following reasoning analogue to that in §2, and in view of the fact that the total energy $H$ of a point particle reduces to the kinetic energy $T$ and that $t$ may be replaced by $s$, we introduce the form $\omega$ that shall serve for the expression Eq. (2.10). It is written:

$$\omega = p_\rho dq^\rho - T ds. \tag{3.7}$$

We now repair to the space $E_9$ comprising the states with points $(q, p, s)$ and trajectories of material points $m$ (See: §2.). The differential equations for these trajectories have the form:

$$\frac{dq^\rho}{Q^\rho} = \frac{dp_\rho}{P_\rho} = \frac{ds}{1}, \quad (\rho = 1,2,3,4),$$

where $Q^\rho, P_\rho$ designate functions of the variables $q^\sigma, p_\sigma, s$.

We introduce now in the affine space $E_9$ the vector field:

$$\vec{V} = (Q^1, Q^2, Q^3, Q^4, P_1, P_2, P_3, P_4, 1).$$

Cartan's Principle is conveyed by the relations:

$$L_V \omega = 0, \quad \underline{\vec{V}|}\omega = 0; \tag{3.8}$$

with which one can obtain the equations of the trajectories of the system $S$. If one expands Eqs. (3.8) (See the Appendix), one gets:

$$\frac{dq^\rho}{\frac{\partial T}{\partial p_\rho}} = \frac{dp_\rho}{-\frac{\partial T}{\partial q^\rho}} = \frac{ds}{1}, \quad (\rho = 1,2,3,4). \tag{3.9}$$

Let us write these equations in the following form:

$$\frac{dq^\rho}{ds} = \frac{\partial T}{\partial p_\rho}, \quad \frac{dp_\rho}{ds} = -\frac{\partial T}{\partial q^\rho},$$

where one has taken account of Eqs. (3.6) and (3.2),

$$\dot{q}^\rho = \frac{1}{2m} g^{\rho\sigma} p_\sigma, \quad \dot{q}_\rho = -\frac{1}{2} m \frac{\partial g_{\sigma\tau}}{\partial q^\rho} \dot{q}^\sigma \dot{q}^\tau.$$

If one differentiates the first of these relations, a calculation that is a bit long but very easy, one then obtains an equation of the following form:

$$\ddot{q}^\rho + \sum_{\sigma\tau} \left\{ \begin{matrix} \sigma\tau \\ \rho \end{matrix} \right\} \dot{q}^\sigma \dot{q}^\tau = 0, \tag{3.10}$$

where $\left\{ \begin{matrix} \sigma\tau \\ \rho \end{matrix} \right\}$ are the well known Christoffel symbols.

Eqs. (3.10) are the equations of geodesics in space-time $E_4$, so that one concludes: Cartan's Dynamics includes the case of point particle in General Relativity.

We note that here the state trajectories have a real existence.

## 4. Systems with nonholonomic constraints[4]

Continuing with the notation and conventions of §2, let us suppose that the coordinates $x_h^\varkappa, t$ $(h = 1,2,\dots,N;\ \varkappa = 1,2,3)$ of a point in space-time $E_3 \times t$ are subject to the constraint of satisfying the following Pfaff system composed of $s < 3N+1$ equations:

$$\Lambda^p = \sum_{h=1}^{N} \left( \sum_{\lambda=1}^{3} a_{h\lambda}^p dx_h^\lambda + a_h^p dt \right) = 0 \quad (p = 1,2,\dots,s), \tag{4.1}$$

where the coefficients are functions of $x_h^\varkappa, t$.

We suppose that Eqs. (4.1), although linear and independent with respect to the differentials $dx_h^\varkappa, dt$, are not necessarily completely integrable. Were they integrable, Eqs. (4.1) would revert to holonomic constraints as treated in §2. Therefore, we shall make a more general hypothesis, namely, that the system of Eqs. (4.1) is an involution with respect to a number $n < 3N+1$ of variables chosen among the $x_h^\varkappa, t$.[5]

Let us change notation now and write Eqs. (4.1) in the following manner:

$$\Lambda^p = dz^p - a_i^p dx^i,$$

where the coefficients designate functions of the variables $x^i, y^t, z^p$ $(i = 1,2,\dots,n; t = 1,2,\dots,q; p = 1,2,\dots s)$, the total number of such variables equals $s+q+n = 3N+1$.

Eqs. (4.1) take the form:

$$\begin{aligned} \Lambda^1 &= dx^1 - a_i^1 dx^i = 0, \\ \Lambda^2 &= dx^2 - a_i^2 dx^i = 0, \\ &\dots\dots\dots\dots\dots\dots \\ \Lambda^s &= dx^s - a_i^s dx^i = 0. \end{aligned} \tag{4.2}$$

In these equations we consider the variables $x^i$ as being independent, and the variables $z^p, t$ as the unknowns.

[4] See: [3, §29].

[5] For a definition of a Pfaff system in involution, see:[2, II, Chapter 5].

In so far as Eqs. (4.1) are in involution with respect to the variables $x^i$, their general integrals can be written as formulas of the form:

$$z^p = \varphi^p(x^1, x^2, \dots, x^n), \quad y^i = \psi^i(x^1, x^2, \dots, x^n), \tag{4.3}$$

where the functions $z^p, y^i$ are independent with respect to the variables $x^i$. By eliminating the variables $x^i$ one obtains relations among the variables $z^p, y^i$, or, in reverting to our previous notation, among the $x^\varkappa_h, t$. We say that the degree of freedom of the nonholonomic system, Eqs. (4.1), equals $n$.

The reasoning above reduces the nonholonomic to holonomic system of constraints as considered above in §2. We may, therfore, apply the method developed above to get the equations of motion for a system with nonholonomic constrains—if we suppose that the material particles, $m_h x^\varkappa_h$ are subject to a force derived from a function $U(x^\varkappa, t)$.

Here we wish to illustrate the method developed above with a simple example.

For this purpose consider a simple particle $s = mx$ $(x = (x^\varkappa) \in E_3)$ subject to constraints defined by a Pfaff equation, i.e.,

$$\Lambda = a_\varkappa dx^\varkappa + a dt = 0, \tag{4.4}$$

where the coefficients are functions of the variables $x^\varkappa, t$. The class $c$ of Eqs. (4.4) can equal 1 or 3 (See:[2, §79]). We prefer the hypothesis that $c = 1$, as in this case Eq. (4.4) leads to a holonomic constraint. If $c = 3$, Eq. (4.4) can be transformed into the following

$$dq^1 - q^2 dq = 0, \tag{4.5}$$

(the canonical form of a Pfaff equation [2, §81]) by means of the analytic transformations

$$x^\varkappa = \psi^\varkappa(q, q^1, q^2), \quad t = \psi(q, q^1, q^2), \tag{4.6}$$

where $q, q^1, q^2$ designate independent variables. It follows from Eq. (4.5) that they may be written:

$$q^1 = \varphi(q), \quad q^2 = \varphi'(q), \tag{4.7}$$

where $\varphi(q)$ designates an arbitrary function of the variable $q$. Thus, one sees that the degree of freedom of a point particle $s$ equals one and that the nonholonomic constraints, Eqs. (4.4), lead to holonomic constraints via relations Eq. (4.7).

Suppose now that the particle $s$ is subject to a force, derived from a function $U(x^\varkappa, t)$, under which the particle moves as specified by the equation

$$m\ddot{x}^\varkappa = \frac{\partial U}{\partial x^\varkappa}, \tag{4.8}$$

and where its kinetic energy is given by

$$T = \frac{1}{2} m \dot{x}^2. \tag{4.9}$$

Consider now the vector $\dot{x} = (\dot{x}^1, \dot{x}^2, \dot{x}^3)$; if one calculates $dx^\varkappa$ and $dt$ starting from Eqs. (4.6) and (4.7), one can convince oneself easily that the quantities $\dot{x}^\varkappa = dx^\varkappa/dt$ and the kinetic energy $T$ can be expressed as functions of the variables $q, t$; as is also true for the function $U(x^\varkappa, t)$ and the vector

$$p = m\dot{x}. \tag{4.10}$$

Thus, Eqs. (4.9) and (4.10) enable expressing kinetic energy as a function of the variables $q, p, t$.

Consider now

$$\omega = pdq - Hdt, \tag{4.11}$$

where the total energy for the particle $s$ equals $H = T - U$, and is also, as we have seen, a function of the variables $q, p, t$. Following the methods discussed above, we introduce the affine space $E_3$ for states $(q, p, t)$ for the particle $s$ and its trajectories. The equation of any particular trajectory can be presented in the form

$$\frac{dq}{Q} = \frac{dp}{P} = \frac{dt}{1}, \tag{4.12}$$

where $Q$ and $P$ are also functions of $q, p, t$. Corresponding to Eqs. (4.12) there exists a vector field $V(Q, P, 1)$ in the state space.

Again, according to Cartan's Principle

$$L_V \omega = 0, \quad \underline{V}| \omega = 0.$$

Expanding, one gets:

$$\frac{dq}{\frac{\partial H}{\partial p}} = \frac{dp}{-\frac{\partial H}{\partial q}} = \frac{dt}{1}; \tag{4.13}$$

so that, we have shown that the motion of a point particle $s$ satisfying the nonholonomic condition, Eqs. (4.4), is determined by Hamilton's equations of motion, Eqs. (4.13).

These results resolve an interesting mechanical problem.

Suppose that on a solid horizontal plate, there is a small wheel of mass $m$ , with a sharp edge and perpendicular to the plate. Let the plate be given orthogonal coordinates $x$ and $y$ for the contact point of the wheel. Suppose that by action of an exterior force parallel to the wheel it moves in its plane. This motion must obey the nonholonomic relations

$$dy - \tan\theta dx = 0$$

(See: [3, p. 39] or [4, p. 227].) where $\theta$ designates the angle of the plane of the wheel with the $x$-axis. It is obvious, while the abstraction is due to the notation, this relation conforms with Eq. (4.5), so that the motion of the wheel is described by Eq. (4.13).

## APPENDIX

Let

$$\omega = a_h dx^h, \quad (h, i, j = 1, 2, \ldots, n), \tag{a1}$$

be a differential form $C^1$ and

$$V = (X^1, X^2, \ldots, X^n)$$

be a vector $C^1$. The operators $\underline{V}|\omega$ and $L_V \omega$ are to be expanded as:

$$\underline{V}|\omega = a_h X^h; \quad L_V \omega = \left( X^i \partial_i a_h - a_i \frac{\partial X^i}{\partial x^n} \right) dx^h \tag{a2}$$

(See [2, pp. 390 & 396]).

1° To apply Eqs. (a1) and (a2) to the form Eq. (1.6 ) and the vector Eq. (1.5), one must take: $n = 7$, $x^4 = t$, $a_1 = p^1$, $a_2 = p^2$, $a_3 = p^3$, $a_4 = a_5 = a_6 = 0$, $a_7 = -H$, $X^1 = Q^1$, $X^2 = Q^2$, $X^3 = Q^3$, $X^4 = P^1$, $X^5 = P^2$, $X^6 = P^3$, $X^7 = 1$.

With these substitutions, Eq. (a2) leads respectively to Eqs. (1.8) to (1.11).

2° Likewise, to apply Eq. (a1) and (a2) to the form Eq. (2.10) and the vector Eq. (2.10), one must take: $n = 2r + i$, $x^\rho = q^\rho$ $(\rho = 1, 2, \ldots, r)$; $x^{r+1} = p_1$, $x^{r+2} = p_2, \ldots$, $x^{2r} = p_r$, $x^{2r+1} = t$, $a_\rho = p_\rho$, $a_{r+1} = a_{r+2} = \ldots = a_{2r} = 0$, $a_{2r+1} = -H$, $V^\rho = Q^\rho$, $V^{r+1} = P_1$, $V^{r+2} = P_2, \ldots$, $V^{2r} = P_r$, $V^{2r+1} = 1$.

With these substitutions, one gets the form Eq. (2.10) and Eqs. (2.8).

3° To get from Eq. (a2) to the formulas of §3, take: $n = 9$, $x^\rho = q^\rho$ $(\rho = 1, 2, 3, 4)$, $x^5 = p_1$, $x^6 = p_2$, $x^7 = p_3$, $x^8 = p_4$, $x^9 = s$, $a_\rho = p_\rho$, $a_5 = a_6 = a_7 = a_8 = 0$, $a_9 = -T$, $V^\rho = Q^\rho$, $V^5 = P_1$, $V^6 = P_2$, $V^7 = P_3$, $V^8 = P_4$, $V^9 = 1$.

4° Finally, to obtain Eq. (4.11), take: $n = 3$, $x^1 = q$, $x^2 = p$, $x^3 = t$; $a_1 = p$, $a_2 = 0$, $a_3 = -H$ ; $V^1 = Q$, $V^2 = P$, $V^3 = 1$.

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