A THEORY OF RELATIVISTIC INTERACTION FOR TWO CHARGED POINT-MASSES.

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Théorie Relativiste de l'Interaction de Deux Particules Chargées
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The purpose of my work[1] has been to find a classical theorectical basis that would furnish the point of departure for research on the differential equations that, in wave mechanics, could represent the interaction of two charged particles. To date, progress in the new mechanics has been obtained by reformulating certain formulas taken from the classical mechanics of point masses into a quantum format. But, from the point of view of relativity, one has had to stop at the problem of a single particle immersed in a given exterior field, because there is no appropriate theory for the motion of two charges subjected to their mutual interaction.

Recall the relativistic equation, which has been introduced in Quantum Mechanics by DIRAC. Let eV be the potential energy of an electron in an electromagnetic field, and let $e\vec{a}/c$ be the corresponding vector potential. Let $p_0, \vec{p}$ be the energy-momentum vector of the electron (we respect the convention that p_0 carries a minus sign). Now, one has an integral of the motion in an equation that expresses the constancy of the scalar mass of the electron:

$$mc = K = \sqrt{\frac{(p_0 - eV)^2}{c^2} - \left(\vec{p} + \frac{e}{c}\vec{a}\right) \cdot \left(\vec{p} + \frac{e}{c}\vec{a}\right)}.$$

The expression constitutes the central function of the canonical equations of motion. By taking ds as the arc length for intervals of $dt, d\vec{x}$, the equations of motion are written then as follows:

$$dt = ds\frac{\partial K}{\partial p_0}, \quad d\vec{x} = ds\frac{\partial K}{\partial \vec{p}},$$
$$dp_0 = -ds\frac{\partial K}{\partial t}, \quad d\vec{p} = -ds\frac{\partial K}{\partial \vec{x}}.$$

It is from this function that DIRAC's theory follows. For our purpose, we must now find an analogous function. We anticipate that for two charged particles, there will be two such functions, that is, one for each particle.

Let us define the system with which we are concerned. The motion of two particles can be represented by two lines in a $(1+3)$-dimensional universe comprising time and space.

The time of the first particle shall be designated by x_0, while its position by x_1, x_2, x_3, i.e., $\vec{x}$, [or as a $(1+3)$-vector: $\mathbf{x}$]; and for the second particle $\mathbf{y}$.

We must establish a correspondence between the instants of each line in such a way that it is unambiguous and invariant under LORENTZ transformations. We shall say that corresponding instants are those connectable with light rays. A light signal departing from a point on the second particle's trajectory, arrives at a point on the first particle's trajectory subsequent in time, which shall be denoted as the 'corresponding instant.' We take it that

[1]Presented at: *l'Assemblée Générale de l'Union International de Physique,* Brussels, 11 July 1931.

x and **y** for the trajectories are given as functions of a parameter in such a way that a particular value of this parameter, denoted as u, enumerates corresponding instants. Also, one has, in all cases, that

$$R^2 = c^2(x_0 - y_0)^2 - (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) = 0,$$

and by differentiating this expression by u, we find that

$$\mathbf{R} \cdot \dot{\mathbf{x}} = \mathbf{R} \cdot \dot{\mathbf{y}},$$

where $\mathbf{R} = \mathbf{x} - \mathbf{y}$, and $\dot{\mathbf{x}} = d\mathbf{x}/du$, and $\mathbf{R} \cdot \dot{\mathbf{x}}$ indicates the $(1+3)$-dimensional inner product.

The retarded LIÉNARD-[WIECHERT] potential at the space-time point **x**, generated by the moving charge e'' found at **y** is determined by:

$$\frac{e''}{4\pi c} \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}}.$$

Reciprocally, the potential anticipated at **y**, generated by the moving charge e' at **x**, is determined by

$$\frac{e'}{4\pi c} \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}}.$$

As I have shown in previous reports[2], one can use these potentials in order to form the term, that in a variation principle for mechanics corresponds to the mutual interaction. The following

$$\frac{e'e''}{4\pi c} \frac{d\mathbf{x} \cdot d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} = \frac{e'e''}{4\pi c} \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{y}}}{\mathbf{R} \cdot \dot{\mathbf{y}}} du,$$

is just this term from which expressions for the mutual forces are extracted.

It is important to remark here, that this term is fully symmetric with respect to the velocities of both charges, because one can replace at one's fancy the factor $\mathbf{R} \cdot d\mathbf{y}$ in the denominator with the factor $\mathbf{R} \cdot d\mathbf{x}$.

On the other hand, one must recognized a certain asymmetry with respect to direction, so to speak, of the action, as the first charge moves under the influence of a retarded potential from the second charge, while it moves under the advanced potential of the first charge. I must admit that one can find this fact inconvenient. It is the price that I propose paying to have a problem that is precise and distinct, and it is needed to get concise formulas.

The masses of the particles are m' and m''. Intervals defined by $d\mathbf{x}$ and $d\mathbf{y}$ are denoted ds' and ds''. The coefficients that appear in

$$ds'^2 = \sum_{i,j} g_{ij} dx^i dx^j = c^2 dx_0^2 - d\vec{x} \cdot d\vec{x} = dx_0 dx^0 + \sum_k^3 dx_k dx^k,$$

are constants, and henceforth, those which satisfy

$$\sum_j g_{j,i} dx_j = dx_i,$$

are designated with a lower index.

The equations of motion are captured by the variational principle

$$0 = \delta \int_1^2 \left\{ -m'c\,ds' - m''c\,ds'' + \frac{e'e''}{4\pi c} \frac{d\mathbf{x} \cdot d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} \right\} = \delta \int_1^2 N du.$$

[2]'Wederkeerigheid in de werking van geladen deeltjes,' *Physica*, **9,** 33 (1929); 'Ein invarianter Variationssatz für die Bewegung meherer elektrisher Massenteilchen,' *Z. f. Phys.* **58**, 386 (1929).

To abbreviate notation, let

$$\frac{e'e''}{4\pi c}=\gamma,\quad \frac{1}{\mathbf{R}\cdot d\dot{\mathbf{x}}}=\frac{1}{\mathbf{R}\cdot d\dot{\mathbf{y}}}=\rho.$$

Now, one finds for the components of the energy-momentum vector:

$$\mathbf{p_x}=\frac{\partial N}{\partial \dot{\mathbf{x}}},\quad \mathbf{p_y}=\frac{\partial N}{\partial \dot{\mathbf{y}}}.$$

It is here that one encounters a peculiar trait. We have already noted that one is free to write either $\mathbf{R}\cdot\dot{\mathbf{x}}$ or $\mathbf{R}\cdot\dot{\mathbf{y}}$ in the denominator or the electromagnetic term; thus, if one uses $\mathbf{R}\cdot\dot{\mathbf{x}}$, then one gets a contribution

$$-\gamma\rho^2(\dot{\mathbf{x}}\cdot\dot{\mathbf{y}})\mathbf{R},$$

for the energy-momentum of the first particle. If, on the other hand, one writes $\mathbf{R}\cdot\dot{\mathbf{y}}$, then one gets the energy-momentum for the motion of the second particle. As a $(1+3)$-dimensional vector it has a direction following the light ray which connects the corresponding instants of the two particles. There is no reason to attribute it to one particle rather than the other. One may attribute it, to a sort of immaterial radiation field, to aether so to speak. Nevertheless, aether does not enter at all into our action integral. This is why we may distribute the portions of the total energy-momentum between the two particles in the proportions $\alpha,\ \beta,\ (\alpha+\beta=1)$, where we shall determine the values later at our convenience. For the moment, we write for the dynamical variables

$$\mathbf{p_x}=m'c\frac{d\mathbf{x}}{ds'}+\gamma\rho\dot{\mathbf{y}}-\alpha\gamma\rho^2(\dot{\mathbf{x}}\cdot\dot{\mathbf{y}})\mathbf{R},$$
$$\mathbf{p_y}=m''c\frac{d\mathbf{y}}{ds''}+\gamma\rho\dot{\mathbf{x}}-\beta\gamma\rho^2(\dot{\mathbf{x}}\cdot\dot{\mathbf{y}})\mathbf{R}.$$

Now, we seek the expressions that give the scalar masses m' and m'' in the relative coordinates and their dynamical variables.

If we let

$$m'c\frac{du}{ds'}=\mu\quad m''c\frac{du}{ds''}=\mu'',$$

we get

$$m'^2c^2=\mu'^2\frac{ds'^2}{du^2}=\mu'^2\dot{\mathbf{x}}^2,\quad m''^2c^2=\mu''^2\dot{\mathbf{y}}^2.$$

Let us seek now $\dot{\mathbf{x}}^2$ and $\dot{\mathbf{y}}^2$ as functions of $\mathbf{p_x}$ and $\mathbf{p_y}$. This task is facilitated by the fact that the scalar product of the factors, of $\mathbf{p_x}$ and $\mathbf{p_y}$, which contain γ, that is to say, from which the mutual interaction follows, cancel out.

$$\left(\frac{\mathbf{p_x}}{\mu'}\right)^2=\dot{\mathbf{x}}^2+(\dot{\mathbf{x}}\cdot\dot{\mathbf{y}})\frac{2\gamma\rho}{\mu'}\left\{\beta-\alpha\frac{\gamma\rho}{\mu'}\right\}+\dot{\mathbf{y}}\frac{\gamma^2\rho^2}{\mu'^2},$$
$$\left(\frac{\mathbf{p_x}\cdot\mathbf{p_y}}{\mu'\mu''}\right)=\dot{\mathbf{x}}^2\frac{\gamma\rho}{\mu''}+(\dot{\mathbf{x}}\cdot\dot{\mathbf{y}})\left\{1-\beta\frac{\gamma\rho}{\mu''}-\alpha\frac{\gamma\rho}{\mu'}\right\}+\dot{\mathbf{y}}^2\frac{\gamma\rho}{\mu''},$$
$$\left(\frac{\mathbf{p_y}}{\mu''}\right)=\dot{\mathbf{x}}^2\frac{\gamma^2\rho^2}{\mu''^2}+(\dot{\mathbf{x}}\cdot\dot{\mathbf{y}})\frac{2\gamma\rho}{\mu''}\left\{\alpha-\beta\frac{\gamma\rho}{\mu''}\right\}+\dot{\mathbf{y}}^2.$$

As for the quantity ρ, note that it appears here either divided by μ' or μ''.

Thus, multiplying the momenta by the action ray $\mathbf{R}$, one gets:

$$\mathbf{R}\cdot\mathbf{p_x}=\frac{\mu'}{\rho}+\gamma,$$

such that:

$$\frac{\rho}{\mu'} = \frac{1}{\mathbf{R}\cdot\mathbf{p_x}-\gamma}, \quad \frac{\rho}{\mu''} = \frac{1}{\mathbf{R}\cdot\mathbf{p_y}-\gamma}.$$

Now it is here that we make use of the liberty to set the proportion α/β. We let

$$\beta - \alpha\frac{\gamma\rho}{\mu'} = \alpha - \beta\frac{\gamma\rho}{\mu''} = k,$$

from which we get

$$\alpha = \frac{\mu'(\mu''+\gamma\rho)}{2\mu'\mu''+\gamma\rho(\mu'+\mu'')} = \frac{1}{2} + \frac{\frac{1}{2}(\mu'-\mu'')\gamma\rho}{2\mu'\mu''+\gamma\rho(\mu'+\mu'')}, \quad \beta = 1-\alpha,$$

and

$$k = \frac{\mu'\mu''-\gamma^2\rho^2}{2\mu'\mu''+\gamma\rho(\mu'+\mu'')}.$$

The values of α, β, and k are invariant under LORENTZ transformation.

This makes obtaining expressions for $\dot{\mathbf{x}}^2, \dot{\mathbf{x}}\cdot\dot{\mathbf{y}}$ and $\dot{\mathbf{y}}^2$ in terms of $\mathbf{p_x}^2, \mathbf{p_x}\cdot\mathbf{p_y}$ and $\mathbf{p_y}^2$ much simpler. One finds:

$$\dot{\mathbf{x}}^2 = \frac{\left(\frac{\mathbf{p_x}}{\mu'}\right)^2 - \frac{\gamma\rho}{\mu'}\left(\frac{\mathbf{p_x}\cdot\mathbf{p_y}}{\mu'\mu''}\right)}{1-\frac{\gamma^2\rho^2}{\mu'\mu''}}, \quad \dot{\mathbf{y}}^2 = \frac{\left(\frac{\mathbf{p_y}}{\mu''}\right)^2 - \frac{\gamma\rho}{\mu''}\left(\frac{\mathbf{p_x}\cdot\mathbf{p_y}}{\mu'\mu''}\right)}{1-\frac{\gamma^2\rho^2}{\mu'\mu''}}.$$

Finally, the sought relations for the masses are:

$$m'^2c^2 = \frac{\mathbf{p_x}\cdot[\mathbf{p_y}-\frac{\gamma\rho}{\mu''}\mathbf{p_y}]}{1-\frac{\gamma^2\rho^2}{\mu'\mu''}}, \quad m''^2c^2 = \frac{\mathbf{p_y}\cdot[\mathbf{p_x}-\frac{\gamma\rho}{\mu'}\mathbf{p_x}]}{1-\frac{\gamma^2\rho^2}{\mu'\mu''}}.$$

The square brackets in these expressions conform with the expressions for a single mass for which we have the translation in DIRAC's theory.

The relations $\gamma\rho/\mu'$ and $\gamma\rho/\mu''$ give us the relations of the mass of electrostatic energy of the respective particles. They are very small. On sees how the mutual action enters in the first instance in the denominators as a correction term.

Briefly, to conclude, we note the importance of these functions for the canonical equations of motion. The variation principle can be taken in the form given by MAUPERTUIS as a principle of least action. We denote $\mathbf{Q}$ as the canonical coordinates corresponding to $\mathbf{x}$, and $\mathbf{P}$ as canonical momenta corresponding to $\mathbf{p_x}$, so that MAUPERTUIS' principle in a relativistic sense takes the form:

$$\delta\int \mathbf{P}\cdot d\mathbf{Q} = 0,$$

where the variation admitted for $\mathbf{P}$ and $\mathbf{Q}$ must satisfy certain constraints:

$$m'c = F(\mathbf{P},\mathbf{Q}), \quad m''c = G(\mathbf{P},\mathbf{Q}),$$

and, in this case, as well as the definition:

$$R^2 = \mathbf{R}\cdot\mathbf{R} = 0.$$

Thus, the variation integral, after an integration by parts, can be written:

$$\int \delta\mathbf{P}\cdot d\mathbf{Q} - \int d\mathbf{P}\cdot\delta\mathbf{Q} = 0,$$

where one takes it, in view of the rules of the calculus of variations considering the imposed constraints:

$$d\mathbf{Q} = d\chi\frac{\partial F}{\partial\mathbf{P}} + d\lambda\frac{\partial G}{\partial\mathbf{P}} + d\nu\frac{\partial R^2}{\partial\mathbf{P}},$$

$$d\mathbf{P} = -d\chi\frac{\partial F}{\partial \mathbf{Q}} - d\lambda\frac{\partial G}{\partial \mathbf{Q}} - d\nu\frac{\partial r^2}{\partial \mathbf{Q}},$$

where χ, λ, ν enter as LAGENDRE's indeterminant multipliers, which can be functions of the integration parameter u.

In this case, these equations become:

$$d\mathbf{x} = ds'\frac{\partial F}{\partial \mathbf{p_x}} + ds''\frac{\partial G}{\partial \mathbf{p_x}},$$

$$d\mathbf{p} = -ds'\frac{\partial F}{\partial \mathbf{x}} - ds''\frac{\partial G}{\partial \mathbf{x}} - d\nu\frac{\partial R^2}{\partial \mathbf{x}},$$

of which one sees quickly that they are rational generalizations for the canonical equations of motion.

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