EIGENTIME IN CLASSICAL AND QUANTUM MECHANICS

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Die Eigenzeit in der klassischen und in der Quantenmechanik
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ABSTRACT. In §1 it is shown, that in the Hamiltonian formulation of nonquantum relativistic mechanics the eigentime τcan be used as the independent variable if a new form of the Lagrangian is employed. In §2, eigentime will be introduced into the DIRAC equation, and a means of integrating this equation based thereon introduced. With the aid of this method, the CAUCHY problem will be treated. In addition, a generalization and simplification of PAULI's application of the WENTZEL-BRILLOUINschen technique to the DIRAC equation will be introduced. In §3 the mixed density in the positron theory will be expressed in terms of the fundamental solution (solution élémentaire) and the RIEMANN function of the DIRAC equation.

1. CLASSICAL MECHANICS

§1.1 Let L^0 be the usual LAGRANGE function from which the relativistic equations of motion for a test charge in an external field are to be derived. L^0 is thought to be

$$L^0 = -mc^2\sqrt{1-\beta^2} - \frac{e}{c}(\vec{r}\cdot\vec{A}) + e\Phi, \tag{1.1}$$

where

$$\beta^2 = \frac{\vec{r}'\cdot\vec{r}'}{c^2}, \tag{1.2}$$

and where primes indicate differentiation with respect to time. If now one introduces the eigentime:

$$\tau = \int_{t^0}^{t}\sqrt{1-\beta^2}dt, \tag{1.3}$$

then the action integral\\\\

$$S = \int_{t^0}^{t} L^0 dt, \tag{1.4}$$

for which variation should yield the equation of motion, can be written in the form:

$$S = \int_{0}^{\tau} L^0 \frac{dt}{d\tau} d\tau. \tag{1.5}$$

Here, however, as the upper limit τ depends on the form of the orbit, its variation too must be taken into account; the consequence is that eigentime τ cannot be considered an independent (non variated) variable in a variation principle in the LAGRANGian function $L^0 dt/d\tau$.

§1.2. One can, however, introduce a new Lagrangian, of the form:s

$$L = \frac{1}{2}m(\dot{r}^2 - c^2\dot{t}^2) - \frac{1}{2}mc^2 - \frac{e}{c}(\dot{\vec{r}}\cdot\vec{A}) + e\dot{t}\Phi, \tag{1.6}$$

where overdots indicate differentiation by the independent variable τ (the quantity τ will turn out later to be identical with the eigentime).

Variation of the integral

$$S = \int_0^\tau L d\tau, \tag{1.7}$$

with the upper limit τ held fixed, then yields the "equations of motion":

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \text{ etc.},$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial t} - \frac{\partial L}{\partial t} = 0. \tag{1.8}$$

As L is not explicitely dependant on τ, these equations have the integral:

$$\dot{r}^2 - c^2\dot{t}^2 = \text{const.} \tag{1.9}$$

Setting the constant equal to $-c^2$ i.e.,

$$\dot{r}^2 - c^2\dot{t}^2 = -c^2, \tag{1.10}$$

yields an equation, according to which the independent variable τ is identical to eigentime; and, Eqs. (1.8) reduce to the usual relativistic equations of motion.

In view of Eq. (1.10), then:

$$L = L^0 \frac{dt}{d\tau}, \tag{1.11}$$

where L^0 has the same meaning as in §1.1. That is, the action integral Eq. (1.7) is numerically identical with Eq. (1.4).

§1.3 With the LAGRANGian function, Eq. (1.6), one can easily obtain HAMILTON's equations of motion, as well as the HAMILTON-JACOBI partial differential equation. The latter has the form:

$$\frac{\partial S}{\partial \tau} + \frac{1}{2m}\left[\left(\nabla S + \frac{e}{c}\vec{A}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t} - e\Phi\right)^2 + m^2c^2\right] = 0. \tag{1.12}$$

In the theory considered herein, r,t play the role of coordinates and τ plays the same role as time in non relativistic mechanics. Therefore, one may directly apply equations from classical mechanics in this theory.

If the action integral, Eq. (1.7), is expressed in the variables $r,t,r^0,t^0;\tau$, one gets:

$$S = S(r,t;r^0t^0;\tau), \tag{1.13}$$

in which τ can be taken as one of the independent variables. The partial differential $\partial S/\partial\tau$ is, as a consequence of the equations of motion, a constant which if set equal to null, 1.e.,

$$\frac{\partial S}{\partial \tau} = 0, \tag{1.14}$$

one gets the condition Eq. (1.10) for the eigentime.

The momentum variables 'conjugate' to r,t, as well as their initial values, can be expressed as partial derivatives as follows:

$$\vec{p}_r = \nabla S; \quad p_t = -H = \frac{\partial S}{\partial t}; \tag{1.15}$$

$$\vec{p}_r^{\,0} = -\nabla_{\vec{r}^0} S; \quad p_t^0 = -H^0 = -\frac{\partial S}{\partial t^0}. \tag{1.16}$$

If Eq. (1.14) is solved for τ and its value put into in Eq. (1.13) for S, one gets the usual action function

$$S = S^*(r,t,r^0,t^0). \tag{1.17}$$

This leads, with Eq. (1.14), obviously to

$$\frac{\partial S^*}{\partial \vec{r}} = \frac{\partial S}{\partial \vec{r}} + \frac{\partial S}{\partial \tau}\frac{\partial \tau}{\partial \vec{r}} = \frac{\partial S}{\partial \vec{r}}. \tag{1.18}$$

The elimination of τ, however, is quite inconvenient. For example, for the problem of an electron in constant electric and magnetic fields, S (with τ) is elementary, while there is no closed form for S^*.

2. Eigentime in the Dirac equation

§2.1 The Dirac wave equation for an electron in a magnetic field can be written:

$$\left\{(\sigma\cdot\vec{P}) + mc\sigma_4 - \frac{T}{c}\right\}\psi = 0. \tag{2.1}$$

Here σ indicates a vector with the Pauli matrices as components, and $\vec{P}$ is the operator

$$\vec{P} = -ih\nabla + \frac{e}{c}\vec{A}, \tag{2.2}$$

for the quantities of motion (canonical momentum) and T is the operator

$$T = ih\frac{\partial}{\partial t} + e\Phi, \tag{2.3}$$

for the kinetic energy of the electron.

The solution ψ of the first order Dirac equation can be given in the form:

$$\psi = \left\{(\sigma\cdot\vec{P}) + mc\alpha_4 + \frac{T}{c}\right\}\Psi, \tag{2.4}$$

where Ψ satisfies the second order differential equation:

$$\left\{\vec{P}^2 + m^2c^2 - \frac{T^2}{c^2} + \frac{eh}{c}(\sigma\cdot\mathfrak{H}) - \frac{ieh}{c}(\sigma\cdot\mathfrak{E})\right\}\Psi = 0, \tag{2.5}$$

that can also be given the form:

$$h^2\Lambda\Psi = 0, \tag{2.6}$$

where the operator Λ is defined as:

$$\Lambda\Psi = -\Box\Psi - \frac{2ie}{hc}\left(\vec{A}\cdot\nabla\Psi + \frac{\Phi}{c}\frac{\partial\Psi}{\partial t}\right) +$$

$$\left\{-\frac{ie}{hc}\left(\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) + \frac{e^2}{h^2c^2}(\vec{A}^2 - \Phi^2) + \frac{m^2c^2}{h^2}\right\}\Psi + \{(\sigma\cdot\mathfrak{H}) - i(\sigma\cdot\mathfrak{E})\}\Psi. \tag{2.7}$$

The solution to Eq. (2.5) can be cast in the form of a definite integral with a help-variable τ:

$$\Psi = \int_C F\,d\tau. \tag{2.8}$$

The equation for Ψ will be satisfied, if F satisfies the differential equation

$$\frac{h}{2m}\Lambda\Psi = ih\frac{\partial F}{\partial \tau}, \tag{2.9}$$

and the integration limits (or the integration path on the complex τ-plane) is so chosen, that the condition:

$$\int_C \frac{\partial F}{\partial \tau} d\tau = F|_C = 0, \tag{2.10}$$

is satisfied.

As will turn out below, the variable τ plays the role of the eigentime. Eq. (2.9) can be seen, therefore, as the *Dirac equation with the eigentime*.

We set

$$F = e^{\frac{i}{\hbar}S} f; \quad \Psi = \int_C e^{\frac{i}{\hbar}S} f \, d\tau, \tag{2.11}$$

where S is the classical action function that satisfies Eq. (1.12). In order to get the equation for f, we note, that

$$\Lambda F = e^{\frac{i}{\hbar}S}\Lambda' f; \quad ih\frac{\partial F}{\partial \tau} = e^{\frac{i}{\hbar}S}\left(ih\frac{\partial f}{\partial \tau} - \frac{\partial S}{\partial \tau} f\right), \tag{2.12}$$

holds, where Λ' comes from Λ if one replaces $\vec{A}$ with $\vec{A} + (e/c)\nabla S$ and Φ with $\Phi - (1/e)\partial S/\partial t$. As a consequence of the differential equation for S, the terms with h in Eq. (2.9) are negligible and one can write an equation for f of the form

$$2m\frac{df}{d\tau} + \left\{\Box S + \frac{e}{c}\left(\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right)\right\} f + \frac{e}{c}\{i(\sigma \cdot \mathfrak{H}) + (\alpha \cdot \mathfrak{E})\} f = ih\Box f. \tag{2.13}$$

Here $df/d\tau$ denotes the full derivative with respect to τ, 1.e.,

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \dot{\vec{r}} \cdot \nabla f + \dot{t}\frac{\partial f}{\partial t}, \tag{2.14}$$

where $\dot{r}, \dot{t}$ are to be understood in their classical meanings, 1.e.,

$$\dot{x} = \frac{1}{m}\left(\frac{\partial S}{\partial x} + \frac{e}{c}A_x\right), \text{ etc.,}$$

$$\dot{t} = -\frac{1}{mc^2}\left(\frac{\partial S}{\partial t} - e\Phi\right). \tag{2.15}$$

§2.2 Eq. (2.13) can be treated advantageously with the method from BRILLOUIN-WENTZEL. Sometimes, for example, with constant electric and magnetic fields, one can find the exact solution in this way. The constant h appears only on the right side of the equation. If the right side is ignored, then one gets the equation:

$$2m\frac{df}{d\tau} + \left\{\Box S + \frac{e}{c}\left(\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right)\right\} f + \frac{e}{c}\{i(\sigma \cdot \mathfrak{H}) + (\alpha \cdot \mathfrak{E})\} f = 0, \tag{2.16}$$

for which the solution is identical with the solution from the usual (not partial) differential equation.

In Eq. (2.16) one can expunge the term with the $\Box$. Let us denote with ρ the absolute value of the fourth order determinant, derived from the second differentials of S with respect to r, t and r^0, t^0 (or the corresponding constants of integration) as follows:

$$\rho = \mathrm{Det}\left\|\frac{\partial^2 S}{\partial x \partial y^0}\right\|. \tag{2.17}$$

The quantity ρ satisfies the 'continuity equation':

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial \vec{r}} \cdot (\rho \dot{\vec{r}}) + \frac{\partial}{\partial t}(\rho \dot{t}) = 0, \tag{2.18}$$

where $\dot{r}, \dot{t}$ have the meaning given above by Eq. (2.15). From this it follows that the quantity $\sqrt{\rho}$ satisfies the equations

$$2m\frac{d\sqrt{\rho}}{d\tau}+\left\{\Box S+\frac{e}{c}\left(\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}\right)\right\}\sqrt{\rho}=0. \tag{2.19}$$

If one sets

$$f=\sqrt{\rho}f^0, \tag{2.20}$$

then f^0 satisfies (in the approximation being used) the following differential equation

$$2m\frac{df^0}{d\tau}+\frac{e}{c}\{i(\sigma\cdot\mathfrak{H})+(\sigma\cdot\mathfrak{E})\}f^0=0. \tag{2.21}$$

For the case in which the fields are constant, one can take it, that f^0 depends only on τ, and not from r,t. Then Eq. (2.21) becomes an ordinary differential equation with constant coefficients. Since in this case ρ too depends only on τ, it follows, that $\Box f=0$, and the approximation, Eq. (2.16), coincides with the exact equation, Eq. (2.13). Thus, using this method one gets an exact solution, which will be given again below in §2.5, p. 11.

The general case of an arbitrary field can be handled in the following manner. With help of Eq. (1.16) for the classical orbits, r,t are to be expressed in terms of τ and inserted into $\mathfrak{H}$ and $\mathfrak{E}$. The coefficients in Eq. (2.21) will be functions of τ alone. In the therewith derived system of ordinary differential equations with variable coefficients, $\vec{p}^0$ and p_t^0 are to be replace with Eqs. (1.16). The result will be, finally, the sought solution for the system of equations.

If one replaces S with S^* in our formulas (Eq. (1.17), then one gets a generalization of PAULI's results,[1] where the BRILLOUIN-WENTZEL method was first employed for solving the DIRAC equation. PAULI's expressions are considerably more complicated, however, because in his investigations he used the first order instead of second order DIRAC equation.

Additionally, the following is to be noted: if the integral Eq. (2.11) is evaluated using the saddle point method, the values of S under the integral are those where $\partial S/\partial\tau=0$, which is the usual (1.e., without τ) function S^*.

§2.3 The form of the solution of the DIRAC equation derived herein (as a definite integral over the eigentime τ) is particularly convenient for study of the CAUCHY problem (1.e., the initial value problem for ψ).

Let ψ be a solution of the first order DIRAC equation, Eq. (2.1) satisfying the initial conditions:

$$\psi=\psi^0;\quad \text{when } t=t^0. \tag{2.22}$$

In order to determine ψ, it is sufficient to find a solution to the second order DIRAC equation $\Lambda\psi=0$, which satisfies the initial condition

$$\Psi=0;\quad \frac{\partial\Psi}{\partial t}=-\frac{ic}{h}\psi^0=\dot{\Psi}^0\quad \text{for } t=t^0. \tag{2.23}$$

The function Ψ can be formulated as an integral

$$\Psi=\int Q\dot{\Psi}^0\,dV, \tag{2.24}$$

where

$$dV=dr^0=dx^0dy^0dz^0. \tag{2.25}$$

[1]PAULI, W., *Helvetica Physica Acta*, **5**, 179 (1932).

$\dot{\Psi}$ is here a given function of r^0, t^0, and Q is a function of both r, t and r^0, t^0. Let us define ξ as

$$\xi = c^2(t-t^0)^2 - (\vec{r}-\vec{r}^0)^2, \tag{2.26}$$

and define the help function, $\gamma(\xi)$ by:

$$\begin{aligned} \gamma(\xi) &= 1 \text{ for } \xi > 0 \\ \gamma(\xi) &= \tfrac{1}{2} \text{ for } \xi = 0 \\ \gamma(\xi) &= 0 \text{ for } \xi < 0 \end{aligned} \tag{2.27}$$

for which the differential $\gamma'(\xi) = \delta(\xi)$, 1.e., the DIRAC delta function.

The quantity Q in the integral Eq. (2.24) is an improper function of the form

$$Q = R\gamma(\xi) + R^*\delta(\xi), \tag{2.28}$$

where R and R^* are continious functions, and R is the so-called RIEMANN function.

Putting Eq. (2.28) for Q into Eq. (2.24), yields for Ψ a sum of two integrals

$$\Psi = \int R\dot{\Psi}^0 \gamma(\xi) dV + \int R^*\dot{\Psi}^0 \delta(\xi) dV. \tag{2.29}$$

The first integral is a volume integral over V, the sphere:

$$c^2(t-t^0)^2 - (\vec{r}-\vec{r}^0)^2 \geq 0,$$

of radius $r^* = |t-t^0|$ centered at $\vec{r}^0 = \vec{r}$. The second integral is an area integral on the surface of the sphere

$$c^2(t-t^0)^2 - (\vec{r}-\vec{r}^0)^2 = 0.$$

If one eliminates the non-constant and complex functions $\gamma(\xi)$ and $\delta(\xi)$ from Eq. (2.29), then in fact one gets:

$$\Psi = \int_V R\dot{\Psi}^0 dV + \frac{1}{2r^*}\int_S R^*\dot{\Psi}^0 dS, \tag{2.30}$$

Since for $t \to t^0$ the radius r^* of the sphere goes to null, so that obviously $\Psi = 0$ for $t = t^0$. Moreover for $t = t^0$ the time differential of the volume integral vanishes. The surface integral, however, for small $t - t^0 > 0$ equals

$$\frac{1}{2r^*}\int_S R^*\dot{\Psi}^0 dS = 2\pi r^*(R^*\dot{\Psi}^0)_0 = 2\pi c(t-t^0)(R^*\dot{\Psi}^0)_0. \tag{2.31}$$

Thus,

$$\left(\frac{\partial\Psi}{\partial t}\right)_{t=t^0+0} = 2\pi c R_0^*\dot{\Psi}^0, \tag{2.32}$$

where R_0^* is the value of R at $\vec{r} = \vec{r}^0, t = t^0$. The initial condition Eq. (2.23) if we set

$$R_0^* = \frac{1}{2\pi c} \tag{2.33}$$

independent of the the coordinates and time.

The function Ψ must also satisfy the second order DIRAC equation. The function Q, therefore, must also satisfy this equation. It must be true, that

$$\Lambda Q = \Lambda(R\gamma(\xi) + R^*\delta(\xi)) = 0. \tag{2.34}$$

Considering the following equations:

$$\Box\gamma(\xi) = -4\delta(\xi), \tag{2.35}$$

$$\Box\delta(\xi) = 0, \tag{2.36}$$

then carrying out the differentiation indicated in Eq. (2.34) gives terms that contain the factors $\gamma(\xi), \delta(\xi), \delta'(\xi)$. If one defines an operator M by

$$MF \equiv (\vec{r}-\vec{r}^0)\cdot\nabla F + (t-t^0)\frac{\partial F}{\partial t} + \frac{ie}{hc}\{(\vec{r}-\vec{r}^0)\cdot\vec{A} - c(t-t^0)\Phi\}F, \tag{2.37}$$

then Eq. (2.34) can be written

$$\Lambda(R\gamma(\xi) + R^*\delta(\xi)) = (\Lambda R)\gamma(\xi) + \{\Lambda R^* + 4(M+1)R\}\delta(\xi) + 4(MR^*)\delta'(\xi). \tag{2.38}$$

This expression vanishes, if

$$\Lambda R = 0, \tag{2.39}$$

$$(M+1)R = -\frac{1}{4}\Lambda R^*, \tag{2.40}$$

$$MR^* = 0. \tag{2.41}$$

It turns out, that Eq. (2.40) is satisfied on the light cone: $\xi = 0$.

A solution for Eq. (2.41) is easily given. If one takes the integral

$$\chi = \int_{(r^0,t^0)}^{(r,t)} (\vec{A}\cdot d\vec{r} - c\Phi dt), \tag{2.42}$$

along a line joining r^0, t^0 and r, t, then

$$(\vec{r}-\vec{r}^0)\cdot\nabla\chi + (t-t^0)\dot{\chi} = (\vec{r}-\vec{r}^0)\cdot\vec{A} - c(t-t^0)\Phi. \tag{2.43}$$

Thus, the function

$$R^* = \frac{1}{2\pi c}e^{-\frac{ie}{hc}\chi}, \tag{2.44}$$

satisfies both Eqs. (2.41) and (2.33), the equation determining R.

With this value of R^* Eqs. (2.39) and (2.40) give equations to solve for R. These equation can be simplified with the supposition

$$R = \frac{1}{2\pi c}e^{-\frac{ie}{hc}\chi}R'. \tag{2.45}$$

This substitution has the effect that the potential $\vec{A}, \Phi$ is replace by

$$\vec{A}' = \vec{A} - \nabla\chi; \quad \Phi' = \Phi + \frac{1}{c}\frac{\partial\chi}{\partial t}. \tag{2.46}$$

If $\vec{A}, \Phi$ satisfy

$$\Box\vec{A} = 0; \ \Box\Phi = 0; \ \nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0, \tag{2.47}$$

then so do $\vec{A}', \Phi'$; moreover, the following relation, derived from Eq. (2.43), also holds:

$$(\vec{r}-\vec{r}^0)\cdot\vec{A} - c(t-t^0) = 0. \tag{2.48}$$

The new potential is uniquely determined by the fields. If one denotes by a double overline the 'average' between the points $\vec{r}^0, t^0$ and $\vec{r}, t$ of f as computed by:

$$\overline{\overline{f}} = 2\int_0^1 f(\vec{r}^0 + (\vec{r}-\vec{r}^0)u,\, t + (t-t^0)u)u\,du, \tag{2.49}$$

then, the following also hold:

$$\vec{A} = -\frac{1}{2}[(\vec{r}-\vec{r}^0)\times\overline{\overline{\mathfrak{H}}}] - \frac{1}{2}c(t-t^0)\overline{\overline{\mathfrak{E}}},$$

$$\Phi' = -\frac{1}{2}(\vec{r}-\vec{r}^0)\cdot\overline{\overline{\mathfrak{E}}}. \tag{2.50}$$

After the substitution of Eq. (2.45), 1.e., after the introduction of the new potentials, Eqs. (2.39) and (2.40) take the following form:

$$\Lambda' R' = 0, \tag{2.51}$$

$$(L+1)R' = -\frac{1}{4}\Lambda' 1 = -\frac{1}{4}\left\{\frac{m^2c^2}{h^2} + \frac{e^2}{h^2c^2}(\vec{A}'^2 - \Phi'^2)\right\} - \frac{1}{4}\frac{e}{hc}\{(\sigma\cdot\mathfrak{H}) - i(\sigma\cdot\mathfrak{E})\}, \tag{2.52}$$

where L is defined by:

$$Lf = (\vec{r}-\vec{r}^0)\cdot\nabla f + (t-t^0)\dot{f}, \tag{2.53}$$

(it could have been denoted M', as it was obtained from M with the new potentials).

Eq. (2.52) delivers not only the value of $(L+1)R'$ for $\xi = 0$, rather also the value of R' for $\xi = 0$. If one considers a function $f(\vec{r}, t)$ as a function of both the quotient $(x-x^0):(y-y^0):(z-z^0):c(t-t^0)$ and the quantity ξ, and if one assumes that for $\xi \to 0$ also $\xi(\partial f/\partial\xi) \to 0$, then the value of $(L+1)f$ for $\xi = 0$ is determined from the value of f for $\xi = 0$, and visa versa.

Let us consider an equation

$$(L+p) = \varphi(\vec{r}, t), \tag{2.54}$$

where p is a positive whole number. A solution of this equation is:

$$f(\vec{r}, t) = \int_0^1 \varphi(\vec{r}^0 + (\vec{r}-\vec{r}^0)u,\, t^0 + (t-t^0)u)u^{p-1}du. \tag{2.55}$$

The only regular solution for the equation $(L+p)f = 0$ in the neighborhood of $\vec{r} = \vec{r}^0$, $t = t^0$ is $f = 0$. For positive p the function f is determined following Eqs. (2.54) and (2.55). If one sets φ in Eq. (2.54) equal to the right side of Eq. (2.52), then one gets, with help of Eq. (2.55), the value of the RIEMANN function on the light cone.

§2.4 As results from the classical study from HADAMARD[2], The RIEMANN function for a hyperbolic differential equation is closely related to its fundamental solution (solution élémentaire). The function $1/r$ is an example of a fundamental solution for the LAPLACE equation; just as

$$\frac{1}{\sqrt{c^2(t-t^0)^2 - (x-x^0)^2 - (y-y^0)^2}},$$

[2] HADAMARD, J., 'Le problème de CAUCHY et les équations aux dérivées partielles linéairs hyperboliques,' (Hermann, Paris, 1932).

is a fundamental solution for the wave equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{c}\frac{\partial^2 u}{\partial t^2}.$$

The fundamental solution to a hyperbolic differential equation has a singularity on the characteristic cone, that depends on the form of the equation. When there are an odd number of independent variables, the fundamental solution is determined uniquely. On the other hand, if the number of independent variables is even, then there exists a infinite number of fundamental solutions; these solutions have a logarithmic singularity for which the coefficient of the logarithm is the RIEMANN function. A fundamental solution can also be found for parabolic differential equations; they can be obtained from the hyperbolic or elliptical cases by 'boundary transgressions.' An elementary example for the parabolic equation is the function

$$u=\frac{1}{\sqrt{y}}e^{-\frac{x^2}{4y}},$$

that is the fundamental solution for the equation $\partial^2 u/\partial x^2=\partial u/\partial y$.

Following these introductory comments, let us turn to the DIRAC equation. The RIEMANN function for the second order DIRAC equation can be put in the form of an integral over the eigentime τ:

$$R=\int F\,d\tau, \tag{2.56}$$

where F represents the fundamental solution of the DIRAC equation with eigentime:

$$\frac{h^2}{2m}\Lambda F=ih\frac{\partial F}{\partial \tau}. \tag{2.57}$$

In this equation, x,y,z,t,τ are the independent variables; and, since this is an odd number, there is a unique fundamental solution.

We seek to find the solution now in the neighborhood of the singular point $\tau=0$. For this purpose we use the Ansatz introduced above:

$$F=e^{\frac{i}{h}S}f, \tag{2.58}$$

and expand both S and f in series of powers of τ. The function S satisfies the HAMILTON-JACOBI equation, Eq. (1.12), with eigentime. Putting the series

$$S=\frac{S_{-1}}{\tau}+S_0+S_1\tau+S_2\tau^2+\cdots, \tag{2.59}$$

into this equation, gives directly:

$$(\nabla S_{-1})^2-\frac{1}{c^2}\left(\frac{\partial S_{-1}}{\partial t}\right)^2=2mS_{-1}. \tag{2.60}$$

Thus, we can set:

$$S_{-1}=\frac{1}{2}m[(\vec{r}-\vec{r}^0)^2-c^2(t-t^0)^2]=-\frac{1}{2}m\xi. \tag{2.61}$$

With this value of S_{-1}, the equation for S_0 becomes:

$$(\vec{r}-\vec{r}^0)\cdot\nabla S_0+(t-t^0)\frac{\partial S_0}{\partial t}=-\frac{e}{c}\{(\vec{r}-\vec{r}^0)\cdot\vec{A}-c(t-t^0)\Phi\}. \tag{2.62}$$

Up to the factor of $-e/c$, this equation agrees with that for χ , 1.e., Eq. (2.43). Thus, one gets:

$$S_0 = -\frac{e}{c}\chi = -\frac{e}{c}\int_{\vec{r}^0,t^0}^{\vec{r},t}(\vec{A}\cdot d\vec{r} - c\Phi dt). \tag{2.63}$$

Finally, the equation for S_1 is:

$$(L+1)S_1 = -\frac{1}{2m}\left\{m^2c^2 + \frac{e^2}{c^2}(\vec{A}'^2 - \Phi'^2)\right\}, \tag{2.64}$$

wherein L is given by Eq. (2.53). The solution of this equation is obtained with the help of Eq. (2.55). Further terms of the series Eq. (2.59) are determined by equations of the form:

$$(L+p)S_p = \varphi_p, \quad (p = 2,3,\ldots), \tag{2.65}$$

where φ_p is known, so long as the preceeding term is known. In this way we get the full solution for S :

$$S = -\frac{m\xi}{2\tau} - \frac{e}{c}\chi - \frac{1}{2}mc^2\tau - \frac{e^2\tau}{2mc^2}\int_0^1(\vec{A}'^2 - \Phi'^2)du + \ldots \tag{2.66}$$

In the field free case this formula yields the exact solution.

Eq. (2.16) can be solved analogously:

$$f = \frac{C}{\tau}\left\{1 - \frac{e\tau}{2mc}\int_0^1[i(\sigma\cdot\mathfrak{H}) + (\sigma\cdot\mathfrak{E})]du + \ldots\right\}. \tag{2.67}$$

Now we must investigate how to select the integration path for the integral:

$$R = \int e^{\frac{i}{h}S} f d\tau, \tag{2.68}$$

so that it yields the RIEMANN function. Eq. (2.68) evidently satisfies the second order DIRAC equation. In order that it coincide with the RIEMANN function, either of the conditions, Eq. (2.40) or (2.52) must be met. We shall seek to show, that that is precisely the case if the integration path is a small circle about the point $\xi = 0$ in the complex τ-plane. We observe to begin, that for $\xi = 0$ the action function S has no poles, so that the point $\tau = 0$ is not a substantial singular point for the integrand. Thus, the integral can be evaluated simply by it residue at the pole $\tau = 0$.

In the vicinity of $\tau = 0$, the integrand equals (for $\xi = 0$)):

(2.69)

$$F = e^{-\frac{ie}{hc}\chi}\left[1 - \frac{imc^2}{2h}\tau - \frac{ie^2\tau}{2hmc^2}\int_0^1(\vec{A}'^2 - \Phi'^2)du - \frac{e\tau}{2mc}\int_0^1(i(\sigma\cdot\mathfrak{H}) + (\sigma\cdot\mathfrak{E}))du + \ldots\right].$$

From Eq. (2.45) we get:

$$R = \frac{1}{2\pi c}e^{-\frac{ie}{hc}\chi}R', \tag{2.70}$$

with

$$R' = 2\pi^2 C\frac{hc}{m}\int_0^1\left[\frac{m^2c^2}{h^2} + \frac{e^2}{h^2c^2}(\vec{A}'^2 - \Phi'^2) + \frac{e}{hc}((\sigma\cdot\mathfrak{H}) - i(\sigma\cdot\mathfrak{E}))\right]du. \tag{2.71}$$

Eq. (2.52) is satisfied if the constant C is set equal to:

$$C = -\frac{m}{8\pi^2 hc}. \tag{2.72}$$

With this value of C, Eq. (2.68) really is the RIEMANN function.

With a suitable selection of integration path one can get the fundamental solution for the second order DIRAC equation from Eq. (2.68).

This fundamental solution for the equation $\Lambda U = 0$, has the form:

$$U = \frac{1}{\pi i}\left[R \lg|\xi| + \frac{R^*}{\xi}\right] + U^*, \tag{2.73}$$

where R and R^* have their earlier meaning [R is the RIEMANN function, R^* is the expression Eq. (2.44)]. The function U^* in Eq. (2.73), in the neighborhood of $\xi = 0$, is regular; it is subjected only to the condition, that the whole expression Eq. (2.73) satisfies the equation $\Lambda U = 0$. This condition obviously is insufficient to determine U^* uniquely; indeed, there are various fundamental solutions, distinguished one from another by the value of their regular part, and which can be derived from the integral Eq. (2.68) by various selections of integration path. The ambiguity of the fundamental solutions corresponds to the fact, that the number of independent variables for the eigentime-free DIRAC equation is even.

§2.5 As an example, let us examine the motion of an electron in constant electric and magnetic fields and calculate for this case the RIEMANN function. To simplify the calculation we limit ourselves to case of parallel fields along the z-axis.

If the potentials are set to

$$A_x = -\frac{1}{2}Hy; \quad A_y = -\frac{1}{2}Hx; \quad A_z = 0; \quad \Phi = -Ez, \tag{2.74}$$

then the Lagrangian function, Eq. (1.6), corresponding to the classical equations of motion is easily determined. If the expression for the action integral, Eq. (1.7), is formed, then it is found, that

$$S = S_0 - \frac{1}{2}mc^2\tau + \frac{eE}{4c}[(z-z^0)^2 - c^2(t-t^0)^2]\coth\frac{eE\tau}{2mc} + \frac{eH}{4c}[(x-x^0)^2 + (y-y^0)^2]\cot\frac{eH\tau}{2mc}, \tag{2.75}$$

where S_0 is the τ independent expression

$$S_0 = -\frac{e}{c}\chi = -\frac{eE}{2}(z+z^0)(t-t^0) - \frac{eH}{2c}(x^0y - y^0x). \tag{2.76}$$

Considering the Eq. (2.13) for f, leads easily to the insight, that this equation can be satisfied by a function f dependant only on τ, so that $\Box f = 0$. Then, Eq. (2.13) reduces to (2.16) (already transformed into Eq. (2.21)). The determinant ρ is, in in this case, equal to

$$\rho = \frac{\text{const.}}{\sin^2\frac{eH\tau}{2mc}\sinh^2\frac{eE\tau}{2mc}}, \tag{2.77}$$

and the solution to Eq. (2.21) would be:

$$f^0 = \exp\left[-\frac{ie}{2mc}\sigma_z H\tau - \frac{e}{2mc}\sigma_z E\tau\right]. \tag{2.78}$$

If the constant factor in f is determined according to Eq. (2.72), the the expression for f becomes:

$$f = -\frac{m}{8\pi^2 hc}\cdot\left(\frac{eH}{2mc}\right)\left(\frac{eE}{2mc}\right)\cdot\frac{f^0}{\sin\frac{eH\tau}{2mc}\sinh\frac{eE\tau}{2mc}}. \tag{2.79}$$

With this value for f, the integral, Eq. (2.68), i.e.,

$$R = \int e^{\frac{i}{\hbar}S} f\,d\tau,$$

when evaluated over a small circle about $\tau = 0$, gives the RIEMANN function for the considered problem.

In the special case for which there are no fields, one gets:

$$f = -\frac{m}{8\pi^2 hc}\frac{1}{\tau^2}; \quad S = -\frac{m\xi}{2\tau} - \frac{1}{2}mc^2\tau, \tag{2.80}$$

and therefrom:

$$R = -\frac{m}{8\pi^2 hc}\int e^{-\frac{im\xi}{2h\tau}-\frac{imc^2\tau}{2h}}\frac{d\tau}{\tau^2} = \frac{m}{4\pi h\sqrt{\xi}}J_1\left(\frac{mc}{h}\sqrt{\xi}\right), \tag{2.81}$$

while the quantity R^* has the constant value $1/2\pi c$. In the presence of fields the function Q of the general theory equals:

$$Q = -\frac{m}{4\pi h\sqrt{\xi}}J_1\left(\frac{mc}{h}\sqrt{\xi}\right)\gamma(\xi) + \frac{1}{2\pi c}\delta(\xi). \tag{2.82}$$

3. Application to positron theory

The fundamentals of DIRAC's formulation[3] of the theory of positrons proposes the notion of a "mixed density" for the distribution of electrons into positive and negative energies.

DIRAC considered mixed densities of two types: namely R_1 and R_F;

$$(\vec{r},t,\zeta|R_1|\vec{r}^0,t^0,\zeta^0) = \sum_{oc.}\psi(\vec{r},t,\zeta)\psi^*(\vec{r}^0,t^0,\zeta) - \sum_{unoc.}\psi(\vec{r},t,\zeta)\psi^*(\vec{r}^0,t^0,\zeta), \tag{3.1}$$

$$(\vec{r},t,\zeta|R_F|\vec{r}^0,t^0,\zeta^0) = \sum_{oc.}\psi(\vec{r},t,\zeta)\psi^*(\vec{r}^0,t^0,\zeta) + \sum_{unoc.}\psi(\vec{r},t,\zeta)\psi^*(\vec{r}^0,t^0,\zeta). \tag{3.2}$$

In these formulas $\psi(\vec{r},t,\zeta)$ denotes the wave function of an electron dependant on the coordinates $\vec{r}$, the time t and spin ζ (eigenvalue index). The sums are over an index (state number) which is here suppressed. The first sum is over all occupied states, the second over unoccupied states.

DIRAC calculated, by directly summing the expressions, Eqs. (3.1) and (3.2) for the field free case; and, then investigated their singularities on the light cone. Then he constructed an analogues expression for the case of a realistic field and determined their singularities by the stipulation, that these expression satisfy the wave equation and that they go over to the previous expression for vanishing fields.

We seek now to show that these quantities, Eqs. (3.1) and (3.2), are precisely those, that arise in CAUCHY's theory of hyperbolic differential equations.

Let us consider first Eq. (3.2). The quantity R_F with respect to the variables ζ and ζ^0 can be written as a matrix:

$$(\vec{r},t|R_F|\vec{r}^0,t^0) = R_F(\vec{r},t). \tag{3.3}$$

This expression, considered as a function of $\vec{r}$ and t, satisfies the DIRAC wave equation:

$$\left\{(\sigma\cdot\vec{P}) + mc\sigma_4 - \frac{T}{c}\right\}R_F = 0, \tag{3.4}$$

and reduces, for $t = t^0$, to the unit operator:

$$R_F = \delta(\vec{r}-\vec{r}^0), \;\; \text{for} \;\; t = t^0. \tag{3.5}$$

Eqs. (3.4) and (3.5) uniquely determine the function R_f. One need not, therefore, calculate the same thing by direct summation, the function R_F can be more easily determined as the solution to CAUCHY's problem.

[3] DIRAC, P. M. A., *Proc. Cambr. Phil. Soc.* **30**, 150 (1934).

The function Q, determined above, Eq. (2.26), satisfies the second order DIRAC equation and the initial condition

$$Q = 0; \quad \frac{\partial Q}{\partial t} = \delta(\vec{r} - \vec{r}^0) \quad \text{FOR} \quad t = t^0. \tag{3.6}$$

From this it follows, that the expression

$$R_1 = -\frac{ic}{h}\left\{(\sigma\cdot\vec{P}) + mc\sigma_4 + \frac{T}{c}\right\}Q, \tag{3.7}$$

with

$$Q = R\gamma(\xi) + R^*\delta(\xi), \tag{3.8}$$

satisfies Eqs. (3.4) and (3.5).

The DIRAC function R_F is thereby expressed in terms of the RIEMANN function.

Concerning the other DIRAC function, R_1, it can be expressed in the same manner by the fundamental solution, U, similar to R_F using Q. We have:

$$R_1 = -\frac{ic}{h}\left\{(\sigma\cdot\vec{P}) + mc\sigma_4 + \frac{T}{c}\right\}U, \tag{3.9}$$

with

$$U = \frac{1}{\pi i}\left[R\ln|\xi| + \frac{R^*}{\xi}\right] + U^*. \tag{3.10}$$

Eq. (3.9) can be considered the fundamental solution to the first order DIRAC equation.

An unambiguous determination of R_1 is possible only with help of initial conditions, made with physical assumptions. One could, for example, demand that for $t = t^0$, all negative energy states of the electron are occupied and all positive states unoccupied. As is to be seen from the original definition of R_1, Eq. (3.1), in the (assumed negative) kernel of the operator for $t = t^0$ must go over to the sign of the kinetic energy. If one solves the DIRAC equation with this initial condition, one gets the value of the mixed density R_1 for all $t > t^0$.

Finally, there is a question of physical interpretation of the mixed density to address. Which terms in this expression have physical meaning? DIRAC, as well as HEISENBERG[4], suggest considering the physical part to be that obtained by subtracting the singularities from the total expression. This scheme can not be taken as correct as it is far too arbitrary. We believe that only the uncertainty principle can be accepted as a reliable criterion for deciding this question. For the considered problem the demands of the uncertainty principle are to be understood as follows: Only those terms can have physical meaning which remain finite for $h \to 0$ and are on the light cone, i.e., $\xi = 0$,;remaining terms are to be disregarded as meaningless. This assumption is verified by study of the polarization of the vacuum, which has been considered by various authors (See:[5].) The additional terms are calculated by these authors form a series in powers of h. This situation indicates the applicability to this problem of the method from BRILLOUIN-WENTZEL discussed in §2.

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