# A RELATIVISTIC LAGRANGIAN FOR MULTIPLE CHARGED POINT-MASSES

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*A translation of:*
**Ein invarianter Variationssatz für die Bewegung mehrerer electrischer Massenteilshen**
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ABSTRACT. A LAGRANGian for multiple, charged, point-masses that is invariant under LORENTZ transformations is presented. It is distinguished by employing both retarded and advanced potentials so that motion is fully time symmetric. Also, the resulting form of conservation principles for momentum and energy are given and their consequences discussed.

## 1. INTRODUCTION

Quantum mechanics for single particles was formulated, by employing the correspondence principle, such that its structure parallels the HAMILTONian formalism of classical physics. With respect to multiple interacting particles, however, no complete quantized formulation has been found that is invariant under LORENTZ transformations. For this task, in fact, even a classical foundation is absent.

It would be, therefore, of use to introduce and develop a proposal for a pre-quantum dynamics for this purpose,[1] that could serve as the foundation for a quantum extention in the form of a HAMILTONian variational principle that takes account of mutual interaction of multiple particles without introducing fields.[2]

## 2. RELATIVISTIC POINT-PARTICLE MECHANICS

To relativize the usual formulation of a variational principle, namely:

$$\delta \int (T-U)dt = 0,$$

where $T$ is the kinetic energy and $U$ the potential energy, as is well known, in stead of $Tdt$, one uses:

$$-mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} dt = -mc\,ds,$$

where $m$ is the mass, $v$ the velocity of the particle, and $c$ the speed of light.

To convert the expression $Udt$ to an invariant scalar, one recognizes that the potential energy is just the time-component of a 4-vector whose spacial part is the negative of momentum $-\mathbf{P}$, which, analogously to energy components, one can denote as "potential momentum." Thus, one may write in place of $Udt$:

$$-Udt + \mathbf{P}\cdot d\mathbf{x} = Udt + P_x dx + P_y dy + P_z dz.$$

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[1] FOKKER, A. D. *Physica* **9**, 33 (1929).

[2] A field formulation was recently proposed by HEISENBERG, W. and PAULI, W., *Z. Phys.* **56** 1 (1929).

When considering the motion of a charge, the covariant potential energy-momentum vector is given by the product of the charge with the field potential, where the 3-vector potential part is to be divided by $-c$.

Let us consider the potential at the point of interest $X$, with space-time coordinates

$$x^0, x^1, x^2, x^3 \Leftrightarrow t, x, y.z \Leftrightarrow t, \mathbf{x},$$

as affected by a charge $e$, with space-time coordinates $w^i$, $i = 0,1,2,3$ as functions of a parameter, $u$. To find the form of the potential, following LIÉNARD and WIECHERT, identify those source or "affective" space-time points or events from which signals traveling at the speed of light reach the point of application. They satisfy:

$$c^2(x^0 - w^0) - (\mathbf{x} - \mathbf{w})^2 = R^2 = 0,$$

or

$$x^0 - w^0 = \frac{r}{c}, \quad r = \sqrt{(\mathbf{x} - \mathbf{w})^2}.$$

If the velocity of the source charge on the line directed towards the point of application is $v_r$, then the potentials $\varphi$ and $\mathbf{a}$ for that charge can be written:

$$\varphi = \frac{e}{4\pi} \frac{1}{r\left(1 - \frac{v_r}{c}\right)}, \quad \mathbf{a} = \frac{e}{4\pi c} \frac{\frac{d\mathbf{w}}{dw^0}}{r\left(1 - \frac{v_r}{c}\right)}.$$

In order to construct a covariant vector from these expressions, we write:

$$\varphi = \frac{e}{4\pi c} \frac{c^2 dw^0}{c^2(x^0 - w^0)dw^0 - (x^1 - w^1)dw^1 \cdots} = \frac{e}{4\pi c} \frac{dw^0}{(R \cdot d\mathbf{w})},$$

and correspondingly

$$-\frac{\mathbf{a}}{c} = \frac{e}{4\pi c} \cdot \frac{-d\mathbf{w}}{c^2(x^0 - w^0) - (x^1 - w^1)dw^1 \cdots} = \frac{e}{4\pi c} \frac{d\mathbf{w}}{(R \cdot d\mathbf{w})},$$

where, clearly, $(R \cdot d\mathbf{w})$ denotes the 4-vector scalar product of the 'interaction ray' $R$ with the displacement differential $d\mathbf{w}$.

The above expression represents a *retarded* potential, which is due to the charge $e$ located at point $\mathbf{w}$ as this charge would affect another located at the point x.

Let us consider now the trajectories of charges $e_2, e_3, \ldots$ as given by their coordinate functions $y^i(u), z^i(v), \ldots$, as functions of the parameters $u, v, \ldots$, respectively.

To determine the solution, $x^i$, of the equation of motion for charge $e_1$ as it is influenced by the other charges, the following LAGRANGian is to be used :

$$0 = \delta \left[ \int -m_1 e ds - \frac{e_1 e_2}{4\pi c} \int \frac{(dx \cdot dy)}{(R \cdot dy)} - \frac{e_1 e_3}{4\pi c} \int \frac{(dx \cdot dz)}{(S \cdot dz)} - \cdots \right],$$

where $ds = dx_1$ and $R, S, \ldots$ are the 'interaction rays' between the location of $dx_1$ and $dy, dz, \ldots$ of the influencing charges at their locations, where, moreover, the magnitude or modulus of the interaction rays always is to equal zero. See Figure: 2.1.

It is essential to require, that the differential $dx_1$ for the charge $e_1$ is to correspond to the motion of the other charges only under the condition:

$$R^2 = 0, \quad S^2 = 0, \ldots \text{etc.}$$

As a consequence of this stipulation, the following always hold:

$$(R \cdot dy) = (R \cdot dx), \quad (S \cdot dz) = (S \cdot dx), \cdots \text{etc.},$$

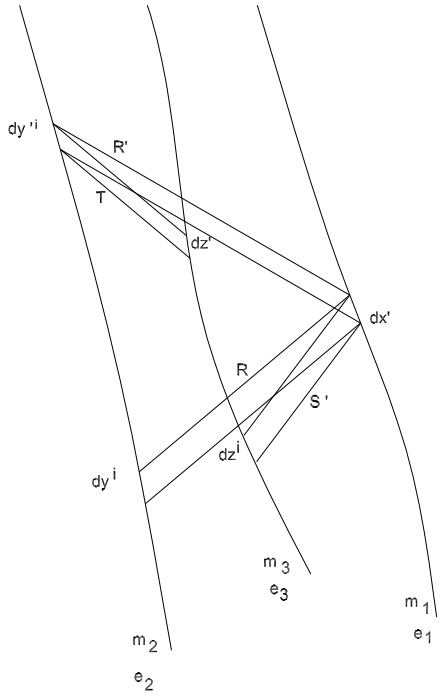


FIGURE 2.1

Were the motions of the charges $e_1, e_3, \ldots$ given *a priori*, and were the task to determine the motion of charge $e_2$ with mass $m_2$, then the differential $dy_2'$ of this motion (with arc-length $ds_2$) would correspond via interaction rays, $R', T', \ldots$ with differential elements on the other trajectories $dx, dz', \ldots$, so that the appropriate LAGRANGian would be:

$$0 = \delta \left[ \int -m_2 e ds_2 - \frac{e_1 e_2}{4\pi c} \int \frac{(dy' \cdot dx)}{(R' \cdot dy)} - \frac{e_2 e_3}{4\pi c} \int \frac{(dy' \cdot dz')}{(T' \cdot dz')} - \cdots \right].$$

Similar expressions obtain for the remaining charges.

## 3. A SYSTEM LAGRANGIAN

At this point, however, it seems auspicious to note that, rather than having a variation for each particle separately, it would be desirable to have a single unified variation for an ensemble as a system. The following fact, however, thwarts this desideratum. The interaction term:

$$-\frac{e_1 e_2}{4\pi c} \int \frac{(dx \cdot dy)}{(R \cdot dy)},$$

accounts for the *retarded* action of $e_2$ on $e_1$, but does not correspond to reciprocal *retarded* influence of $e_1$ on $e_2$. Nevertheless, because $(R \cdot dy) = (R \cdot dx)$, this same interaction term in the form

$$-\frac{e_1 e_2}{4\pi c} \int \frac{(dy \cdot dx)}{(R \cdot dy)},$$

accounts for exactly the *advanced* interaction of $e_1$ on $e_2$.

One may well assert—by no means here for the first time—that the habit of preferring to work only with *retarded* potentials, is simply an arbitrary prejudice. In view of the aim to have a fully reciprocal interaction, it might be taken, that $e_2$ influences $e_1$ to one half via *retarded*, and one half via *advanced* interaction. Such would have as a consequence, that the term in the LAGRANGian accounting for the *retarded* action of the second charge on the first, also accounts for *advanced* action of the first on the second. This in turn, permits using a *single* LAGRANG*ian function* to determine trajectories for all particles of a system, i.e.:

$$0 = \delta\left[\sum\int -m_i c\, ds_i - \sum \frac{e_i e_j}{8\pi c}\left\{\int \frac{(dx\cdot dy)}{(R\cdot dy)} + \frac{e_2 e_3}{4\pi c}\int \frac{(dy'\cdot dx)}{(R'\cdot dx)}\right\}\right],$$

where the first sum is over all particles in the ensemble, and the second is over the particle pairs for which mutual interaction is to be taken into account.

This LAGRANGian is fully invariant under LORENTZ transformations. It makes no use of the notion of 'field' whatsoever. The reciprocity and symmetry of the interaction is complete. It can be said, that it pertains more to a system of motions, than to a system of particles.

The concept of a system of particles would require a certain ordering of the differentials of motion. This ordering can be uniquely specified, but then it would not be invariant; or, it can be made invariant, but then it would not be unambiguous, i.e., it is not possible to define a system of particles that is simultaneously both invariant and unambiguous. Thus, unavoidably, one must choose a system of motions over a system of particles.

## 4. CONSRVATION PRINCIPLES AND CONCLUSIONS

Now we consider a variation of the trajectory of particle with charge $e_1$ for which at each space-time point $\mathbf{x}$, an infinitesimal variation $\delta\mathbf{x}$ shall be introduced into the interaction:

$$-\frac{e_1 e_2}{8\pi c}\delta\int\frac{(dx\cdot dy)}{(R\cdot dy)},$$

so as to calculate the effect of $e_2$ on $e_1$. By this variation, the established correspondence between the differentials $d\mathbf{x}$ and $d\mathbf{y}$ are to be retained: The interaction ray, $R$, is to be a light signal ($|R| = 0$), and the variation $\delta\mathbf{x}$ should imply intrinsically the variation:

$$\delta\mathbf{y} = \frac{d\mathbf{y}}{(R\cdot dy)}(R\cdot\delta x)$$

of the motion of $e_2$. This motion changes nothing, its only effect is to secure the correspondence between the variation differentials, i.e., to assure that $(R\cdot\delta y) = (R\cdot\delta x)$, and therefore also $\delta R^2 = 0$. In addition, the correspondence of the quantities: $(R\cdot dy) = (R\cdot dx)$ remains intact, so that for the variation of the denominator one can take the variation of $R\cdot dx)$.

Thus, we may now write:

$$\delta\frac{(dx\cdot dy)}{(R\cdot dy)} = \frac{(\delta dx\cdot dy) + (dx\cdot\delta dy)}{(R\cdot dy)} - \frac{(dx\cdot dy)\delta(R\cdot dx)}{(R\cdot dy)(R\cdot dx)}.$$

One can integrate this expression by parts and verify that variation at the limits of the integral, vanishes. This leads to an integrand, keeping the value of $\delta\mathbf{y}$ in view:

$$\sum_{i,m}\delta x^i\Big[-d\left\{\frac{dy_i}{(R\cdot dy)} - R_i\frac{(dy\cdot dx)}{(R\cdot dx)(R\cdot dy)}\right\} - R_i\frac{dy^m}{(R\cdot dy)}d\frac{dx_m}{(R\cdot dx)} - dx_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)} + R_i\frac{(dx\cdot dy)^2}{(R\cdot dx)(R\cdot dy)^2}\Big].$$

The expression in the parenthesis gives an increase in kinetic energy and size of the motions (for $i = 0$ or $i = 1,2,3$), in short, the force exerted on the charge $e_1$. One sees that, except from the dependences on velocity that are usually ignored, the force depends partially on accelerations of the source charges, and partially on terms independent of accelerations. The last term corresponds to electrostatic forces and the so-called radiation effects (the accurant and subsequent term).

In our formulations we have also included the advanced influence of $e_2$ in the integral

$$-\frac{e_1e_2}{8\pi c}\int\frac{(dx\cdot dy')}{(R'\cdot dx)} = -\frac{e_1e_2}{8\pi c}\int\frac{(dx\cdot dy')}{(R'\cdot dy')}.$$

Executing the variation in the same way as above while taking account of the fact that $\delta\mathbf{R}' = \delta\mathbf{y}' - \delta\mathbf{x}$, then requires the vanishing of the integral:

$$\begin{aligned}0 = d\left(m_1c\frac{dx_i}{ds_1}\right) + \frac{e_1e_2}{8\pi c}\Big[&d\left\{\frac{dy_i}{(R\cdot dy)} - R_i\frac{(dy\cdot dx)}{(R\cdot dx)(R\cdot dy)}\right\}\\ &+ d\left\{\frac{dy'_i}{(R\cdot dy)} - R'_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy)}\right\}\\ &+ R_i\sum\frac{dy^m}{(R\cdot dy)}d\frac{dx_m}{(R\cdot dx)} + dx_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)} - R_i\frac{(dx\cdot dy)^2}{(R\cdot dx)(R\cdot dy)^2}\\ &+ R'_i\sum\frac{dy'^m}{(R'\cdot dy')}d\frac{dx_m}{(R\cdot dx)} - dx_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')} + R_i\frac{(dx\cdot dy')^2}{(R'\cdot dx)(R'\cdot dy')^2}\Big].\end{aligned}$$

Within the large parenthesis, which, brought to the left side with minus signs would be seen as a covariant vector representing transfer of energy and momentum from $e_2$ to $e_1$, there are some total differentials.

On could interpret them as energy and momentum of $e_1$ which is due to the presence and motion of $e_2$.

For the effect of $e_1$ on the second particle, there is a similar equation also comprised of two components. If we write this equation for the differential $dy'$, (See Figure: 4.1), which via the interaction ray $R' = 0$ is bound with the previously considered differential $dx$ and with the interaction ray $R'' = 0$ is bound with the differential $dx''$, we get:

$$\begin{aligned}0 = d\left(m_1c\frac{dy'_i}{ds_2}\right) + \frac{e_1e_2}{8\pi c}\Big[&-d\left\{\frac{dx_i}{(R'\cdot dx)} - R'_i\frac{(dy'\cdot dx)}{(R'\cdot dx)(R'\cdot dy')}\right\}\\ &+ d\left\{\frac{dx''_i}{(R''\cdot dx'')} - R''_i\frac{(dy'\cdot dx'')}{(R''\cdot dx'')(R''\cdot dy')}\right\}\\ &+ R'_i\sum\frac{dx^m}{(R'\cdot dx)}d\frac{dy'_m}{(R'\cdot dy)} + dy'_i\frac{(dy'\cdot dx)}{(R'\cdot dy')(R'\cdot dx)} - R'_i\frac{(dx\cdot dy')^2}{(R'\cdot dy')(R'\cdot dx)^2}\\ &+ R''_i\sum\frac{dx''^m}{(R''\cdot dx'')}d\frac{dy'_m}{(R''\cdot dy')} - dy'_i\frac{(dy'\cdot dx'')}{(R''\cdot dy')(R''\cdot dx'')} + R''_i\frac{(dy'\cdot dx'')^2}{(R''\cdot dy')(R''\cdot dx'')^2}\Big].\end{aligned}$$

It is important to recognize here, that the third row in this expression together with the last row of the previous expression, constitute a total derivative. Both of them relate to the mutual interaction ray $R'$ between $dx$ and $dy'$. Together they represent

$$\frac{e_1e_2}{8\pi c}d\left\{R'_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy'}\right\}.$$

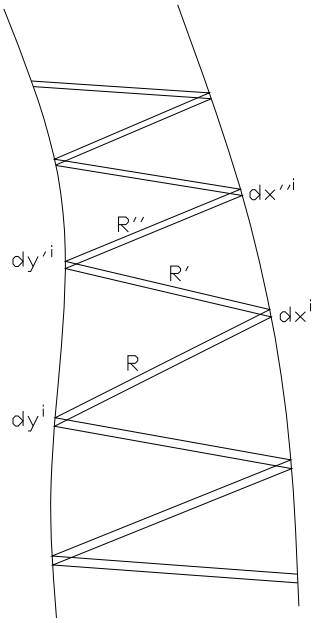


FIGURE 4.1

This expression, in turn, leads the way to an understanding of how the principles of conservation of energy and momentum pertain in this formulation. To make things simple, let us consider only two particles.

We must consider a sum of a progressive zig-zag chain of interaction rays (See: Fig. 2) linking differentials of motion and their equations.

In this sum, each vertex of the zig-zag chain, depending on whether it pertains to the action of $e_1$ on $e_2$, or visa-versa, corresponds to a contribution to the kinetic energy of one or the other particle:

$$d\left(m_1 c\frac{dx_i}{ds_1}\right), \quad \text{or} \quad d\left(m_2 c\frac{dy_i}{ds_2}\right),$$

while for each interaction-ray, $R'^i$ between $dx^i$ and $dy'^i$ there corresponds a contribution to the potential energy of:

$$\frac{e_1 e_2}{8\pi c} d\left\{\frac{dx_i}{(R'\cdot dx)} + \frac{dy'_i}{(R'\cdot dy')} - R'_i\frac{(dx\cdot dy)}{(R'\cdot dx)(R'\cdot dy')}\right\}.$$

If the motion is periodic, then the chain will also be periodic, or very nearly so, and the sum then would extend only over a single period, as it closes back on itself. In this case it constitutes a total differential, so that there is a quantity that remains constant along the chain. If one divides this constant quantity by the number of mutual interactions in a period, one gets the energy, or the momentum, [of the system].

This corraborates the remark made above, that energy and momentum can not be defined for a system of particles, but rather only for a system of mutual motions.

Over and above this fact, when the motion is not periodic, the sum does not close on itself, and therefore, does not yield a total differential. At the ends of the zig-zag chain there will always remain excluded contributions. If they are ignored nevertheless, and the sum is divided by the number of interaction rays, an ever more exact definition of the total energy and momentum can be given as the chain is taken longer and longer; but no matter what, the energy and momentum at a particular moment will remain undefined.

This is the price for excluding fields from the LAGRANGian; yet, this feature does not conflict with quantum mechanics.

*Translator:* A. F. KRACKLAUER 

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