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A geometric proof of no-interaction theorems

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No-interaction theorems are proven using the methods of differential geometry; and, an example of a Hamiltonian yielding relativistic canonical equations of motion with interaction is presented.

I. INTRODUCTION

It is the purpose of this note to present a proof of "no-interaction" theorems using the language of modern differential geometry [1] and Cartan's principle [2] of dynamics. The use of these tools reveals certain facts of the structure of these theorems which are not otherwise evident.

The conclusion of a no-interaction theorem is that a relativistic canonical formulation of dynamics can only describe particles between which there is "no-interaction." Of course, this conclusion depends on the conditions in the hypothesis of the theorem, of which there are two versions. The original version has been proven sequentially by stronger methods, first for two [3], then three [4], and finally N particles by at least three methods [5-7], one of which (Ref. [7]) uses modern differential geometry, but not Cartan's Principle. This version is characterized by the assumption that the dynamics of the system is governed by a scheme parameterized by a single parameter—time. The second version [8], which has been proven heretofore by only one method, is characterized by the assumption that the dynamics is governed by an N parameter scheme.

Below, the no-interaction result is obtained in a new way by adding restrictions to Cartan's Principle that are equivalent to the conditions in the hypotheses of the no-interaction theorems. From this unique vantage point these theorems are reexamined.

II. A GEOMETRIC PROOF

Let (M_N, Ω) be a symplectic manifold [1], where M_N is the Cartesian product of N phase spaces, with fundamental two-form

$$\Omega = d\omega. \tag{1}$$

ω is expressed in terms of the canonical momenta and coordinates as

$$\omega = \sum^{N} (p_1^{\mu} dx_{\mu})_i, \tag{2}$$

(summation convention implied). Recall [2] that a one-form on $M_N \times I$, for example ω':

$$\omega' = \omega - H d\tau, \tag{3}$$

determines a vector field D on $(M_N \times I)$ of the form:

$$D = \sum_{1}^{N} \left(V^{\mu} \frac{\partial}{\partial p^{\mu}} + F_{\mu} \frac{\partial}{\partial x_{\mu}} \right)_i + \frac{\partial}{\partial \tau}, \tag{4}$$

via the stipulation that the exterior product of the pair is zero (this is the formal statement of Cartan's Principle); i.e.,

$$D \wedge d\omega' = 0, \tag{5}$$

such that V and F satisfy then Hamilton's canonical equations:

$$V = \frac{\partial H}{\partial p}; \quad F = -\frac{\partial H}{\partial x}. \tag{6}$$

Theorem (no-interaction): Let (M_n, Ω) be a symplectic manifold, D a vector field on $M_N \times I$, and H a scalar function. Moreover, let I be identified with a one-dimensional subspace fro each configuration space: for example, let

$$\tau = a x_i^1 = b x_j^2 = \cdots = c x_k^N. \tag{7}$$

Then,

$$D \wedge d\omega' = 0, \tag{8}$$

implies that:

$$F_1 = F_2 = \cdots = F_n = 0; \tag{9}$$

i.e., all forces are zero, there is no interaction.

Proof: The identification of I with a one-dimensional subspace of the configuration space of a particle implies that the generator of translations along I, namely:

$$\frac{\partial}{\partial \tau}, \tag{10}$$

is equal to the generators of the translations along the one-dimensional subspace in the configuration space, for example

$$a \frac{\partial}{\partial x_i^1}, \tag{11}$$

where it has been assumed for simplicity, without loss of generality, that the identification has been made with the $i-$th axis in the coordinate frame chosen to describe the configuration space of interest. When this identification is made for each configuration space in M_N, the following equality holds:

$$\frac{\partial}{\partial\tau} = a\frac{\partial}{\partial x_i^1} = b\frac{\partial}{\partial x_j^2} = \cdots = c\frac{\partial}{\partial x_k^N}. \tag{12}$$

If Eq. (12) is put into (8), then the result Eq. (9) follows at once. ■

Theorem (second version): The fundamental assumption of this version is that the dynamics of a system of particles is governed by an N parameter scheme. This assumption is tantamount to redefining Eq. (4) as:

$$D = \sum_1^N \left[\left(V^\mu\frac{\partial}{\partial p^\mu} + F_\mu\frac{\partial}{\partial x_\mu}\right)_i + \frac{\partial}{\partial\tau_i}\right], \tag{13}$$

and Eq. (3) as

$$\omega' = \omega + \sum_1^N H_i d\tau_i. \tag{14}$$

Proof: As above, Eq. (8) is computed to obtain

$$\{H_i, H_l\} = 0, \quad \forall i \neq j, \tag{15}$$

where the brackets are those of Poisson. The independence of the world line follows from the independence of Hamiltonians. ■

III. DISCUSSION AND CONCLUSIONS

The equivalence of the first version, as presented herein, with the previous presentations follows from the demands made of the Hamiltonian in those versions. First, and most naturally, the Hamiltonian is to specify the the dynamics through the Lie Bracket relationships:

$$[x, H] = -V; \quad [p, H] = F. \tag{16}$$

Beyond this, however, the very same Hamiltonian function is also to be the generator of time translations which together with the generators of the spacial translations form a Lie Group with the Poincaré group structure constants. Imposing this group structure is a common way of "relativizing" the space of interest. Demanding that one function yield both sorts of generators at once establishes the identification used above, of translations along τ (the parameter of the transformation specifying the dynamics) and x_i^n (a configuration parameter, most likely t, time, if M_n is a Minkowski space).

The absence of reference to Minkowski structure of the configuration spaces in the above proof shows that the essential feature of no-interaction theorems is not to be found in Special Relativity. The above theorems hold even if the spaces M_n are not Minkowski spaces; the only essential feature needed to obtain the no-interaction result is the identification of the parameter of the transformation generating the dynamics with any of the parameters of the configuration spaces. In the non-relativistic case, if "t" were identified with either x, y, or z (or [non]linear combinations), the no-interaction result would follow, a fact not heretofore widely publicized.

The motivation for the identification of the dynamical parameter with a configuration parameter in the first place comes from the desire to create relativistic quantum theories. In nonrelativistic quantum mechanics time is a scalar parameter while configuration variables are operators. This disparity is at odds with Special Relativity within which time and space parameters are of equal status. Thus, an effort has been made to find a relativistic formulation of dynamics in which the configuration variable "t " (x_4 in Minkowski space) can serve as the independent parameter of the transformation specifying the dynamics and thereby be compatible with quantum theory as presently formulated. It is these efforts which are frustrated by the first version of no-interaction theorems.

The second version is an exploratory attempt to find a structure that will accommodate interaction and be consistent with relativity. As the structure hypothesized in this version is not compatible with quantum theory, it does not appear to have any significance for the construction of relativistic quantum theory; moreover, it also does not accommodate interaction.

If the identification of time and the parameter which governs the dynamics is not made, as it is in the original versions of no-interaction theorems, then Cartan's formulation of dynamics (or any equivalent) accommodates the construction of canonical relativistic theories at will; it is only necessary to find a suitable single parameter Lorentz invariant scalar function $\mathcal{H}$. What is not so easily done, however, is to show that a particular choice of $\mathcal{H}$ leads to calculated trajectories that have observable correspondents. For an example of a Hamiltonian that may describe the electromagnetic interaction, consider the following: Let m_i be the rest mass of the i-th particle and let x_i and $\dot{x}_i$ be defined as follows:/q

$$x_i \equiv (x_i, y_i, z_i, ict_i); \quad \dot{x}_i \equiv dx_i/d\tau. \tag{17}$$

If now the Lagrangian $\mathcal{L}$, where

$$\begin{aligned} \mathcal{L}(x_i(\tau), \dot{x}_i(\tau)) &= \sum_i^N \frac{m_i\dot{x}_i\cdot\dot{x}_i}{2} \\ &\quad - 2\sum_{j\neq i}^N e_i e_j \int_{-\infty}^{\tau} \dot{x}_j \delta\left((x_i(\tau) - x_j(\lambda))^2\right) d\lambda, \end{aligned} \tag{18}$$

is posited (dot products are with respect to the Lorentz metric), then by employing the well-known definition of canonical momentum,

$$\mathcal{P}_i \equiv \frac{\partial\mathcal{L}}{\partial\dot{x}_i}, \tag{19}$$

one deduces the Hamiltonian

$$\begin{aligned} &\mathcal{H}(x_i(\tau)\mathcal{P}_i(\tau)) \\ &= \sum_i^N \left(\mathcal{P}_i(\tau) - 2\sum_{j\neq i}^N e_i e_j \int_{-\infty}^{\tau} \dot{x}_j \delta\left((x_i(\tau) - x_j(\lambda))^2\right) d\lambda\right)^2 / (2m_i). \end{aligned} \tag{20}$$

This Hamiltonian leads to equations of motion which are differential-delay equations of motion coupled together by two and only two interactions, each derived from a Lienard-Wiechert potential. Although Cauchy-type initial data is insufficient to determine a particular solution to these equations, they can be integrated numerically given the orbits between the past and the future of a light cone centered at an arbitrary point as initial data. The results of such a study will be reported elsewhere; the point here is only that Cartan's principle does accommodate canonical relativistic dynamics with interaction if the effort to give time a role distinct from space is abandoned.

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