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A theory of the electromagnetic two-body interaction

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A theory of the electromagnetic two-body interaction is described which leads to equations of motion solvable by local (numerical) integration.

I. INTRODUCTION

Although Newton's action-at-a-distance theory of gravity is similar to Coulomb's Law for charged particles, it was found that its structure is inadequate to describe completely the electromagnetic force because it assumes "instantaneous interaction." The effort to overcome this inadequacy lead to Maxwell's equations which, while adequate for practical applications, are, at the most basic level, beset with an impediment—that they do not lead to a closed set of coupled equations for two or more charged point particles. Instead, one particle is first considered as a current for which Maxwell's equations are solved for the values of the field variables at the location of the second particle whose response is determined by the Lorentz Force Law. Then, the second particle is considered as a current whose fields perturb the motion of the first current. The recalculated first current is used thereafter to compute more accurate values of the field variables, etc. This is continued back and forth to obtain the solutions to the desired degree of accuracy.

Reacting to this situation, Fokker developed a closed formulation for the electromagnetic force by incorporating light-cone interaction into action-at-a-distance mechanics. Essentially he found a Lagrangian which is not merely the sum of individual Lagrangians whose variation yields coupled equations of motion.[1] This Lagrangian, however, produced yet another complexity: It led to simultaneous advanced and retarded interaction for each particle. This feature is problematic on two levels. First, it raises questions of causality because it would mean that the present is always partially conditioned by all of the future, contrary to observation. Secondly, it introduces the calculational complication of precluding the known methods of integrating the equations of motion (this point will be discussed below).

No resolution for the causality difficulties of the *pure* two-particle problem appear to have been proposed; in fact, apparently the only attempt at resolution immerses the problem in a many body universe by invoking radiation absorbers at infinity.[2] Moreover, although integration of the pure two-particle equations has been attempted, thus far the proposed schemes are clearly approximation techniques or useful only in severely restricted circumstances.[3; 4]

This problem continues to be of great interest and is being studied from many perspectives. Some of these are to be found in Ref's. [5–9].

It is the purpose of this article to describe a theoretical formulation of the electromagnetic force whose equations of motion can be integrated by known methods and in which advanced interaction, although not completely eliminated, appears at most as an effect of only limited extent. The essence of this formulation, first presented using Cartan's principle and modern differential geometry[10], will be elaborated herein avoiding abstruse techniques.

II. THE THEORY

The essence of this theory is that it has a single independent parameter which is given no *a priori* physical role (although it has *a posteriori* physical utility); its function is analogous to that of a step counter in a numerical calculation. It does have *a priori* mathematical significance, however, as a dynamical parameter in the sense that it is the independent variable for which the canonical variables are dependent. The only objects with physical significance in this formulation are the world lines; everything else, including the independent parameter, is a mathematical aid to their calculation.

Let $\mathbf{x}_j$ be the Minkowski configuration four-vector with components x_j, y_j, z_j, ict_j of the j-th particle. Let $\mathbf{dx}_j$ be a differential displacement along the j-th particle's orbit. Two such differentials tangent to arbitrary points p and p' on orbits j and k are related to each other by the Lorentz transformation $L(p,\, p',\, j,\, k)$, between the instantaneous rest frames of j and k; i.e., given $\mathbf{dx}_j|_p$, $\mathbf{dx}_k|_{p'}$ is essentially defined by

$$\mathbf{dx}_k|_{p'} = L(p,\, p',\, j,\, k)\mathbf{dx}_j|_p). \tag{1}$$

Thus, the differential of arc length,$(\mathbf{dx}_j \cdot \mathbf{dx}_j)^{1/2}$ is invariant because at each point it satisfies

$$\begin{aligned}(\mathbf{dx}_k|_{p'} \cdot \mathbf{dx}_k|_{p'})^{1/2} =& \ (\mathbf{dx}_j|_p L^\dagger \cdot L\mathbf{dx}_j|_p)^{1/2} \\ =& \quad (\mathbf{dx}_j|_p \cdot \mathbf{dx}_j|_p)^{1/2}.\end{aligned} \tag{2}$$

All such differentials may, therefore be set equal to the common differential $c\,d\tau$, where c is the speed of light and τ is the independent parameter which assumes the units of time; i.e.,

$$c\,d\tau = (\mathbf{dx}_j \cdot \mathbf{dx}_j)^{1/2} = (\mathbf{dx}_k \cdot \mathbf{dx}_k)^{1/2}. \tag{3}$$

Dividing (2.3) by c and rewriting yields

$$d\tau = dt_j \gamma_j^{-1} = dt_k \gamma_k^{-1}, \tag{4}$$

where $\gamma_j^{-1} = \left(1-(v_j/c)^2\right)^{1/2}$ in the customary notation.

Digressing momentarily, observe that a particle's proper time, $\triangle\tau_j$, in this formulation is computed by integration from Eq. (4) to be

$$\triangle\tau_j = \int_{\triangle t_j} \gamma_j^{-1} dt_j. \tag{5}$$

Because the γ_j are not in general equal, it follows that the values of $\triangle\tau_j$ for different particles are also, in general, unequal. Although a single variable is the proper time for each particle, its *values* are not simultaneously (i.e., for equal $t_0 + \triangle t_j$) relevant to each particle.

Continuing, let four-velocities be defined as

$$\mathbf{v}_j := \mathbf{dx}_j/d\tau = \gamma_j(\mathbf{v}_j, ic) := \dot{\mathbf{x}}_j \tag{6}$$

and momenta as $m_j\mathbf{v}_j$, where m_j is the j-th particle's rest mass. With these definitions, the four-vector version of Hamilton's principle

$$\delta \int_{\tau_1}^{\tau_2} \mathcal{L}(\mathbf{x_j}, \mathbf{v_j}, \tau)\mathbf{d\tau} = \mathbf{0}, \tag{7}$$

where (for N (number of particles) =2)

$$\begin{aligned}\mathcal{L} = \quad & \sum_{j=1}^{2} m_j \left(\mathbf{v}_j \cdot \mathbf{v}_j\right)^{1/2} \\ & -2\sum_{k\neq j}^{2} e_j e_k \int_{-\infty}^{\tau} \mathbf{v}_j(\tau)\cdot\mathbf{v}_k(\tau')\delta\left(\left(\mathbf{x}_j(\tau)-\mathbf{x}_k(\tau')\right)^2\right) d\tau',\end{aligned} \tag{8}$$

yields equations of motion coupled by only two interactions. (Because of the upper bound on the integral, not all possible interactions are included.)

There are, however, two forms these equations can take, depending on the character of $\mathbf{x}_j(\tau) - \mathbf{x}_k(\tau)$ for a particular value of τ; case I, when it is space like

$$m_j\ddot{\mathbf{x}}_j = \frac{e_j}{c}\left(\sum_{k\neq j} F_k|_{\text{ret}}\right)^{\mu\upsilon} (\dot{\mathbf{x}}_j)_\upsilon\,, \quad j = 1, 2, \tag{9}$$

and case II when it is time-like ($t_a - t_b > 0$)

$$\begin{aligned} m_a\ddot{\mathbf{x}}_a &= \frac{e_j}{c}\left(F_b|_{\text{ret}} + F_b|_{\text{adv}}\right)^{\mu\upsilon} (\dot{\mathbf{x}}_a)_\upsilon\,, \\ m_b\ddot{\mathbf{x}}_b &= 0 \end{aligned} \tag{10}$$

where

$$F_k^{\mu\upsilon} = 2c\int_{-\infty}^{\tau} \left(\dot{\mathbf{x}}_k^{\upsilon}\partial_\upsilon - \dot{\mathbf{x}}_k^{\mu}\partial_\mu\right)\delta\left(\left(\dot{\mathbf{x}}_j(\tau) - \dot{\mathbf{x}}_k(\tau)\right)^2\right) d\tau' \tag{11}$$

(For $N > 2$, complex combinations of Eqs. (9) and (10) may hold.)

Case I appears to be more natural, each particle proceeds under the retarded influence of the other. Case II is entirely novel; here one particle (b say) has punctured the future light cone of the other (a) so that the further motion of b is is unaffected by a which responds, however, to both retarded and advanced signals from b.[1]

An interesting possibility is that a system might switch back and forth between cases I and II. Consider, for example, two oppositely charged particles, one very massive, the other not. Suppose they are initially constrained such that both their world lines are initially pure time-like up to a time t_0 when they are released. At this point the lighter particle particle would accelerate toward its partner, which, by comparison, would remain virtually stationary. The extention of these world lines into the future beyond t_0 can be computed according to Eq. (9). The world line of the massive particle would continue virtually parallel to the segment preceeding t_0, and filar marks corresponding to increments of τ would be evenly spaced. The world line of the lighter particle would both curve and be extended by increasingly longer increments for each incremental increase of τ. Because it is asymptotically approaching a light like line, where an infinitely long line on the diagram has zero length, filar marks on the light particle's world line appear to be at increasing intervals. This disparity will cause the lighter particle at some point to puncture the future light cone of the heavier particle and the system will pass into the case II regime where the lighter particle is free and its world line straight. It can be shown that eventually the lighter particle will re-emerge and the system again enter the case I regime.

III. COMPARISON WITH FOKKER'S FORMULATION

The features peculiar to this theory can best be delineated by comparison with Fokker's formulation. The most outstanding difference is that Fokker's formulation does not exploit Eq. (3) and therefore employs a separate independent parameter for each particle. Fokker's Lagrangian is not simply the sum of individual Lagrangians patched together in an *ad hoc* manner; he argued that a truly fundamental formulation should proceed from the variation of a *single* system Lagrangian to a set of coupled equations of motion. The La-

[1] *After-the-fact note:* Case II does not exist. Presuming that it did, was based on failure to take into account the fact that Lorentz transforms induce severe anisotropism on Minkowski space, an error that leads many others also into several problems, including asymmetric aging.

grangian $\mathcal{L}_F$,

$$\mathcal{L}_F = \sum_j^N L_j = \sum_j^N m_j \left(\mathbf{v}_j \cdot \mathbf{v}_j\right)^{1/2}$$
$$-2\sum_{k\neq j}^{2} e_j e_k \int_{-\infty}^{+\infty} \mathbf{v}_j(\tau_j)\cdot\mathbf{v}_k(\tau_k)\delta\left((\mathbf{x}_j(\tau_j)-\mathbf{x}_k(\tau_k))^2\right) d\tau_k, \tag{12}$$

satisfies these criteria and leads, by means of the variation

$$\delta\int\sum_j^N L_j d\tau_j = 0, \ \ j = 1,2,\ldots,N, \tag{13}$$

to the equations of motion

$$m_j\ddot{\mathbf{x}}_j(\tau_j) = \frac{e_j}{2c}\sum_{k\neq j}^N \left(F_k|_{\mathrm{ret}} + F_k|_{\mathrm{adv}}\right)^{\mu\upsilon}\left(\dot{\mathbf{x}}_a(\tau_j)\right)_\upsilon ,$$
$$j = 1,2,\ldots N. \tag{14}$$

These equations, however, cannot be integrated by a local procedure as is obvious if one imagines attempting a machine integration of the j-th equation at a given value of τ_j. Such an integration; i.e., a calculation of the an incremental extention of the world line for an incremental increase in τ_j, requires knowledge of the k-th world line on the forward light cone of the j-th particle, which, in order to be computed, requires knowledge of the i-th world line on the forward light cone of the j-th particle, but this portion of this orbit is yet to be computed, etc., *ad infinitum.* In effect, the solution is needed as initial data in order to compute the solution in this way.

Of course, advanced interaction could be precluded by changing the upper limit of integration in Eq. (12) to τ_{ij}, where τ_{ij} is that value of τ_j which includes only the retarded potential from the j-th particle; however, as τ_{ij} would then also appear in Eq. (12), it could be written as the sum of individual Lagrangians and, therefore, would not qualify as a system Lagrangian.

Schemes can be imagined which circumvent this problem by some sort of global approach; i.e., by seeking the whole solution at once. For example, perhaps the solution could be found as the limit of a technique each successive step of which gave a closer approximation to the entire world line. At present, however, such techniques do not appear to have been developed—Eqs. (14) are in general numerically and analytically unsolvable.

Eqs. (9) and (10), on the other hand, can always be integrated by machine because the information needed to compute each incremental increase of any world line in both cases I and II has already been computed. Also by imagining a machine calculation, it is clear that if each particle's world line between the past and the future with respect to the same but otherwise arbitrary light cone is given as initial data, then the system of world lines can be extended by calculation indefinitely into the future or the past. Although this type of initial data is greater that the customary Cauchy data $\{\mathbf{x}(\tau_a), \dot{\mathbf{x}}(\tau_a)\}$, it is a general characteristic of differential-delay equations that Cauchy data are insufficient to determine a particular solution as enough initial data must be given to span the delay.[11; 12]

IV. RADIATION REACTION

Because the classical derivation of the mathematical expressions for radiation reaction employs advanced potentials[13], which this formulation excludes as a persistent feature, a new physical model of radiation reaction is needed.

Assuming that the universe as a whole is electrically neutral, let us take it that a particular charge will induce among all other charges a coincident virtual negative image charge. Radiation reaction is assumed then to be the interaction of a charge with its own induced image. The equations of motion for this system are Eq. (9), where particle 1 is the charge and particle 2 is its image. Solving this system is made easier by the following. One, to first order, $\mathbf{x}_1$ equals $\mathbf{x}_2$ (modulo effects of radiation lag). Two, the interaction from the induced image implodes on the charge as if from an oppositely charged concentric spherical shell. To an accelerated charge, in its own frame, this interaction is identical to that of a pre-counter-accelerated shell, which in turn, is identical to the sign-changed, time-reversed effect of the charge itself; i.e., $F_2|_{\mathrm{ret}}$ equals $F_1|_{\mathrm{adv}}$. With this substitution, Eqs. Eq. (9) can be added to give (note $e_2 = -e_1$)

$$m_a\ddot{\mathbf{x}}_a = \frac{e_a}{2c}\left(F_a|_{\mathrm{ret}} - F_a|_{\mathrm{adv}}\right)^{\mu\upsilon}\left(\dot{\mathbf{x}}_a\right)_\upsilon , \tag{15}$$

This equation is precisely the starting point of Dirac's derivation of an explicit form for the force of radiation reaction, which is not herein reiterated.[14]

V. COMMENTS AND CONCLUSIONS

The Lagrangians Eqs. (8) and (12) both employ a notational gimmick that can lead to confusion. The problem is that in both formulations two types of integrations appear, each with a distinct function. In Eq. (7) the integration on τ and in Eq. (13) the integration on τ_j belong to the variational principle; whereas, the remaining integrations really are superfluous. They are part of a notational gimmick used to express Liénard-Wiechert potential in an elegant form by exploiting the properties of the Dirac delta function.[15] In fact, the delta function can be expanded and the integrations over the dummy variables τ_k in Eq. (12) and τ' in Eq. (8) executed to write these Lagrangians in a more transparent form before executing the variation. This form would preclude confusion regarding the distinct roles of the various τ's and integrations, albeit at a cost in notational elegance.

The structure of differential-difference equations, such as Eq. (9), is such that there is not a unique orbit through each point is phase space. This fact is another facet of the requirement for more than Cauchy initial data. A consequence of this fact is that there is no surface, space-like or otherwise, perpendicular to all orbits whose evolution is regulated by the dynamics such that it could be parameterized by a single variable. This has lead to the belief that a single variable cannot be used to parameterize all individual orbits of a multiparticle system; however, when each orbit is regarded independently, no problems arise for lack of such a surface or other simple correlation between filar marks on world lines.[2]

The essential difference between various formulations of the electromagnetic two-body problem is the selection of interactions. Any formulation in which the the interactions are derived from Liénard-Wiechert potentials is consistent with Maxwell's equations. In this formulation the mathematical formalism selects only retarded interaction except when one particle punctures the future light cone of another to become free while the latter "sees" retarded and advanced interactions. Further study may show, however, that this transition effect cannot occur in realistic (many-body) circumstances. But, if it does occur, it might confirm the validity of this formalism. Confirmation can in principle also be obtained by comparing observed with computed world lines (when the formalism permits), but again many-body or quantum effects probably intervene to make this difficult or ambiguous.

In conclusion, this article describes a formulation for the electromagnetic force whose equations of motion can be integrated by a local scheme (i.e., mechanically) and which reveals a potential novel physical effect manifested by straight segments of world lines in interacting particle. Moreover, this formulation affords new insights into radiation reaction.

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[2] *After-the-fact-note:* Subsequent work has shown that the integrated length (or number of filar marks) between two crossing points of separate orbits are equal, i.e., there is no asymmetric aging.