On the Theory of Quanta

LOUIS-VICTOR DE BROGLIE (1892-1987)

PARIS

A translation of :

RECHERCHES SUR LA THÉORIE DES QUANTA

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Preface to German translation

In the three years between the publication of the original French version, [as translated to English below,] and a German translation in 1927[1], the development of Physics progressed very rapidly in the way I foresaw, namely, in terms of a fusion of the methods of Dynamics and the theory of waves. M. EINSTEIN from the beginning has supported my thesis, but it was M. E. SCHRÖDINGER who developed the propagation equations of a new theory and who in searching for its solutions has established what has become known as "Wave Mechanics." Independent of my work, M. W. HEISENBERG has developed a more abstract theory, "Quantum Mechanics", for which the basic principle was foreseen actually in the atomic theory and correspondence principle of M. BOHR . M. SCHRÖDINGER has shown that each version is a mathematical transcription of the other. The two methods and their combination have enabled theoreticians to address problems heretofore unsurmountable and have reported much success.

However, difficulties persist. In particular, one has not been able to achieve the ultimate goal, namely a undulatory theory of matter within the framework of field theory. At the moment, one must be satisfied with a statistical correspondence between energy parcels and amplitude waves of the sort known in classical optics. To this point, it is interesting that, the electric density in Maxwell-Lorentz equations may be only an ensemble average; making these equations non applicable to single isolated particles, as is done in the theory of electrons. Moreover, they do not explain why electricity has an atomised structure. The tentative, even if interesting, ideas of MIE are thusly doomed.

Nonetheless, one result is incontestable: NEWTON's Dynamics and FRESNEL's theory of waves have returned to combine into a grand synthesis of great intellectual beauty enabling us to fathom deeply the nature of quanta and open Physics to immense new horizons.

Paris, 8 September 1927

[1] *Untersuchungen zur Quantentheorie,* BECKER, W. (trans.) (Aka. Verlag., Leipzig, 1927).

Introduction

History shows that there long has been dispute over two viewpoints on the nature of light: corpuscular and undulatory; perhaps however, these two are less at odds with each other than heretofore thought, which is a development that quantum theory is beginning to support.

Based on an understanding of the relationship between frequency and energy, we proceed in this work from the assumption of existence of a certain periodic phenomenon of a yet to be determined character, which is to be attributed to each and every isolated energy parcel, and from the Planck-Einstein notion of proper mass, to a new theory. In addition, Relativity Theory requires that uniform motion of a material particle be associated with propagation of a certain wave for which the phase velocity is greater than that of light (CHAPTER 1).

For the purpose of generalising this result to nonuniform motion, we posit a proportionality between the momentum world vector of a particle and a propagation vector of a wave, for which the fourth component is its frequency. Application of Fermat's Principle for this wave then is identical to the principle of least action applied to a material particle. Rays of this wave are identical to trajectories of a particle (CHAPTER 2).

The application of these ideas to the periodic motion of an electron in a Bohr atom leads then, to the stability conditions of a Bohr orbit being identical to the resonance condition of the associated wave (CHAPTER 3). This can then be applied to mutually interacting electrons and protons in hydrogen atoms (CHAPTER 4).

The further application of these general ideas to EINSTEIN's notion of light quanta leads to several very interesting conclusions. In spite of remaining difficulties, there is good reason to hope that this approach can lead further to a quantum and undulatory theory of Optics that can be the basis for a statistical understanding of a relationship between light-quanta waves and MAXWELL's formulation of Electrodynamics (CHAPTER5̃).

In particular, the study of scattering of X and γ-rays by amorphous materials, reveals just how advantageous such a reformulation of electrodynamics would be (CHAPTER 6).

Finally, we see how introduction of phase waves into Statistical Mechanics justifies the concept of existence of light quanta in the theory of gases and establishes, given the laws of black body radiation, how energy parcellation between atoms of a gas and light quanta follows.

Historical survey

From the 16th to the 20th centuries. The origins of modern science are found in the end of the 16th century, as a consequence of the Renaissance. While Astronomy rapidly developed new and precise methods, an understanding of equilibrium and motion from the study of dynamics and statics improved only slowly. As is well known, NEWTON was first to unify Dynamics to a comprehensive theory which he applied to gravity and thereby opened up other new applications. In the 18th and 19th centuries generations of mathematicians, astronomers and physicists so refined NEWTON's Mechanics that it nearly lost its character as Physics. This whole beautiful structure can be extracted from a single principle, that of MAUPERTUIS, and later in another form as HAMILTON's Principle of least action, of which the mathematical elegance is simply imposing.

Following successful applications in acoustics, hydrodynamics, optics and capillary effects, it appeared that Mechanics reigned over all physical phenomena. With somewhat more difficulty, in the 19th century the new discipline of Thermodynamics was also brought within reach of Mechanics. Although one of the main fundamental principles of thermodynamics, namely conservation of energy, can easily be interpreted in terms of mechanics, the other, that entropy either remains constant or increases, has no mechanical clarification. The work of CLAUSIUS and BOLTZMANN, which is currently quite topical, shows that there is an analogy between certain quantities relevant to periodic motions and thermodynamic quantities, but has not yet revealed fundamental connections. The imposing theory of gases by MAXWELL and BOLTZMANN, as well as the general statistical mechanics of GIBBS and BOLTZMANN, teach us that, Dynamics complimented with probabilistic notions yields a mechanical understanding of thermodynamics.

Since the 17th century, Optics, the science of light, has interested researchers. The simplest effects (linear propagation, reflection, refraction, etc.) that are nowadays part of Geometric Optics, were of course first to be understood. Many researchers, principally including DESCARTES and HUYGENS, worked on developing fundamental laws, which then FERMAT also succeeded in doing with the principle that carries his name, and which nowadays is usually called the principle of least action. HUYGENS propounded an undulatory theory of light, while NEWTON , calling on an analogy with the theory of material point dynamics that he created, developed a corpuscular theory, the so-called "emission theory", which enabled him even to explain, albeit with contrived hypothesis, effects nowadays considered wave effects (e.g., NEWTON's rings).

The beginning of the 19th century saw a trend towards HUYGEN's theory. Interference effects, made known by YOUNG's experiments, were difficult or impossible to explain in terms of corpuscles. Then FRESNEL developed his beautiful elastic theory of light propagation, and NEWTON's ideas lost credibility irretrievably.

A great successes of FRESNEL's theory was the clarification of the linear propagation of light, which, along with the Emission theory, was extraordinarily simple to explain. We note, however, that when two theories, seemingly on entirely different basis, can clarify with equal facility an experimental result, then one should ask if a difference is real or an artifact of accident or prejudice. In FRESNEL's age such a question was unfashionable and so the corpuscular theory was ridiculed as naive and rejected.

In the 19th century there arose a new physics discipline of enormous technical and theoretical consequence: the study of electricity. We need not remind ourselves of contributions by VOLTA, AMPERE, LAPLACE, FARADAY, etc. For our purposes it is noteworthy, that MAXWELL mathematically unified results of his predecessors and showed that all of optics can be regarded as a branch of electrodynamics. HERTZ, and to an even greater extent LORENTZ, extended MAXWELL's theory; LORENTZ introduced discontinuous electric charges, as was experimentally already demonstrated by J. J. THOMSON. In any case, the basic paradigm of that era retained FRESNEL's elastic conceptions, thereby holding optics apart from mechanics; although, many, even MAXWELL himself, continued to attempt to formulate mechanical models for the aether, with which they hoped to explain all electromagnetic effects.

At the end of the century many expected a quick and complete final unification of all Physics.

The 20th century: Relativity and quantum theory. Nevertheless, a few imperfections remained. Lord KELVIN brought attention to two dark clouds on the horizon. One resulted from the then unsolvable problems of interpreting MICHELSON's and MORLEY's experiment. The other pertained to methods of statistical mechanics as applied to black body radiation; which while giving an exact expression for distribution of energy among frequencies, the Rayleigh-Jeans Law, was both empirically contradicted and conceptually unreal in that it involved infinite total energy.

In the beginning of the 20th century, Lord KELVIN's clouds yielded precipitation: the one led to Relativity, the other to Quantum Mechanics. Herein we give little attention to aether interpretation problems as exposed by MICHELSON and MORLEY and studied by LORENTZ and FITZGERALD, which were, with perhaps incomparable insight, resolved by EINSTEIN—a matter covered adequately by many authors in recent years. In this work we shall simply take these results as given and known and use them, especially from Special Relativity, as needed.

The development of Quantum Mechanics is, on the other hand, of particular interest to us. The basic notion was introduced in 1900 by MAX PLANCK. Researching the theoretical nature of black body radiation, he found that thermodynamic equilibrium depends not on the nature of emitted particles, rather on quasi elastic bound electrons for which frequency is independent of energy, a so-called Planck resonator. Applying classical laws for energy balance between radiation and such a resonator yields the Rayleigh Law, with its known defect. To avoid this problem, PLANCK posited an entirely new hypothesis, namely: *Energy exchange between resonator (or other material) and radiation takes place only in integer multiples of* $h\nu$*, where h is a new fundamental constant.* Each frequency or mode corresponds in this paradigm to a kind of atom of energy. Empirically it was found: $h = 6.545 \times 10^{-27}$ erg-sec. This is one of the most impressive accomplishments of theoretical Physics.

Quantum notions quickly penetrated all areas of Physics. Even while deficiencies regarding the specific heat of gases arose, Quantum theory helped EINSTEIN, then NERST and LINDEMANN, and then in a more complete form, DEBYE, BORN and KARMANN to develop a comprehensive theory of the specific heat of solids, as well as an explanation of why classical statistics, i.e., the Dulong-Petit Law, is subject to certain exceptions and finally why the Rayleigh Law is restricted to a specific range.

Quanta also penetrated areas where they were unexpected: gas theory. BOLTZMANN's methods provided no means to evaluate certain additive constants in the expression for entropy. In order to enable NERST's methods to give numerical results and determine these additive constants, PLANCK, in a rather paradoxical manner, postulated that the phase space volume of each gas molecule has the value h^3.

The photoelectric effect provided new puzzles. This effect pertains to stimulated ejection by radiation of electrons from solids. Astoundingly, experiment shows that the energy of ejected electrons is proportional to the frequency of the incoming radiation, and not, as expected, to the energy. EINSTEIN explained this remarkable result by considering that radiation is comprised of parcels each containing energy equal to $h\nu$, that is, when an electron adsorbs energy equal to $h\nu$, and the ejection itself requires energy equal to w, then the election has an amount of energy equal to$h\nu - w$. This law turned out to be correct. Somehow EINSTEIN instinctively understood that one must consider the corpuscular nature of light and suggested the hypothesis that radiation is parcelled into units of $h\nu$. As this notion conflicts with wave concepts, at first most physicists rejected it. Serious objections from, among others, LORENTZ and JEANS, EINSTEIN rebutted by pointing to the fact that this same hypothesis, i.e., discontinuous light, yields the correct black body law. The international Solvay conference in 1911 was devoted totally to quantum problems and resulted in a series of publications supporting EINSTEIN by POINCARÈ, which he finished shortly before his death.

In 1913 BOHR's theory of atom structure appeared. He took it, along with RUTHERFORD and VAN DER BROEK, that atoms consist of positively charged nuclei surrounded by an electron cloud, and that a nucleus has N positive charges, each of 4.77×10^{-10}esu. and that its number of accompanying electrons is also N, so that atoms are neutral. N

is the atomic number that also appears in MENDELEJEFF's chart. To calculate optical frequencies for the simplest atom, hydrogen, BOHR made tow postulates:

1.) Among all conceivable electron orbits, only a small number are stable and somehow determined by the constant h. In CHAPTER 3, we shall explicate this point.

2.) When an electron changes from one to another stable orbit, radiation of frequency ν is absorbed or emitted. This frequency is related to a change in the atom's energy by $|\delta\varepsilon| = h\nu$.

The great success of BOHR's theory in the last 10 years is well known. This theory enabled calculation of the spectrum for hydrogen and ionised helium, the study of X-rays and MOSELEY's Law, which relates atomic number with X-ray data. SOMMERFELD, EPSTEIN, SCHWARTZSCHILD, BOHR and others have extended and generalised the theory to explain the Stark Effect, the Zeemann Effect, other spectrum details, etc. Nevertheless, the fundamental meaning of quanta remained unknown. Study of the photoelectric effect for X-rays by MAURICE DE BROGLIE, γ-rays by RUTHERFORD and ELLIS have further substantiated the corpuscular nature of radiation; the quantum of energy, $h\nu$, now appears more than ever to represent real light. Still, as the earlier objections to this idea have shown, the wave picture can also point to successes, especially with respect to X-rays, the prediction of VON LAUE's interference and scattering (See: DEBYE, W. L. BRAGG, etc.). On the side of quanta, H. A. COMPTON has analysed scattering correctly as was verified by experiments on electrons, which revealed a weakening of scattered radiation as evidenced by a reduction of frequency.

In short, the time appears to have arrived, to attempt to unify the corpuscular and undulatory approaches in an attempt to reveal the fundamental nature of the quantum. This attempt I undertook some time ago and the purpose to this work is to present a more complete description of the successful results as well as known deficiencies.

CHAPTER 1

The Phase Wave

1.1. The relation between quantum and relativity theories

One of the most important new concepts introduced by Relativity is the inertia of energy. Following EINSTEIN, energy may be considered as being equivalent to mass, and all mass represents energy. Mass and energy may always be related one to another by

$$\text{energy} = \text{mass} \times c^2, \tag{1.1.1}$$

where c is a constant known as the "speed of light", but which, for reasons delineated below, we prefer to denote the "limit speed of energy." In so far as there is always a fixed proportionality between mass and energy, we may regard material and energy as two terms for the same physical reality.

Beginning from atomic theory, electronic theory leads us to consider matter as being essentially discontinuous, and this in turn, contrary to traditional ideas regarding light, leads us to consider admitting that energy is entirely concentrated in small regions of space, if not even condensed at singularities.

The principle of inertia of energy attributes to every body a proper mass (that is a mass as measured by an observer at rest with respect to it) of m_0 and a proper energy of m_0c^2. If this body is in uniform motion with velocity $v = \beta c$ with respect to a particular observer, then for this observer, as is well known from relativistic dynamics, a body's mass takes on the value $m_0/\sqrt{1-\beta^2}$ and therefore energy $m_0c^2/\sqrt{1-\beta^2}$. Since kinetic energy may be defined as the increase in energy experienced by a body when brought from rest to velocity $v = \beta c$, one finds the following expression:

$$E_{\text{kin.}} = \frac{m_0c^2}{\sqrt{1-\beta^2}} - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \tag{1.1.2}$$

which for small values of β reduces to the classical form:

$$E_{\text{kin.}} = \frac{1}{2}m_0v^2. \tag{1.1.3}$$

Having recalled the above, we now seek to find a way to introduce quanta into relativistic dynamics. It seems to us that the fundamental idea pertaining to quanta is the impossibility to consider an isolated quantity of energy without associating a particular frequency to it. This association is expressed by what I call the 'quantum relationship', namely:

$$\text{energy } = h \times \text{frequency} \tag{1.1.4}$$

where h is PLANCK's constant.

The further development of the theory of quanta often occurred by reference to mechanical 'action', that is, the relationships of a quantum find expression in terms of action instead of energy. To begin, PLANCK's constant, h , has the units of action, ML^2T^{-1}, and this can be no accident since relativity theory reveals 'action' to be among the "invariants" in physics theories. Nevertheless, action is a very abstract notion, and as a consequence of much reflection on light quanta and the photoelectric effect, we have returned to statements on energy as fundamental, and ceased to question why action plays a large role in so many issues.

The notion of a 'quantum' makes little sense, seemingly, if energy is to be continuously distributed through space; but, we shall see that this is not so. One may imagine that, by cause of a meta law of Nature, to each portion of energy with a proper mass m_0, one may associate a periodic phenomenon of frequency ν_0, such that one finds:

$$h\nu_0 = m_0c^2. \tag{1.1.5}$$

The frequency ν_0 is to be measured, of course, in the rest frame of the energy packet. This hypothesis is the basis of our theory: it is worth as much, like all hypotheses, as can be deduced from its consequences.

Must we suppose that this periodic phenomenon occurs in the interior of energy packets? This is not at all necessary; the results of §1.3 will show that it is spread out over an extended space. Moreover, what must we understand by the interior of a parcel of energy? An electron is for us the archetype of isolated parcel of energy, which we believe, perhaps incorrectly, to know well; but, by received wisdom, the energy of an electron is spread over all space with a strong concentration in a very small region, but otherwise whose properties are very poorly known. That which makes an electron an atom of energy is not its small volume that it occupies in space, I repeat: it occupies all space, but the fact that it is indivisible, that it constitutes a unit.[1]

Having supposed existence of a frequency for a parcel of energy, let us seek now to find how this frequence is manifested for an observer who has posed the above question. By cause of the Lorentz transformation of time, a periodic phenomenon in a moving object appears to a fixed observer to be slowed down by a factor of $\sqrt{1-\beta^2}$; this is the

[1] Regarding difficulties that arise when several electric centres interact, see CHAPTER 4 below.

famous clock retardation. Thus, such a frequency as measured by a fixed observer would be:

$$\nu_1 = \nu_0\sqrt{1-\beta^2} = \frac{m_0c^2}{h}\sqrt{1-\beta^2}. \tag{1.1.6}$$

On the other hand, since the energy of a moving object equals $m_0c^2/\sqrt{1-\beta^2}$, this frequency according to the quantum relation, Eq. (1.1.4), is given by:

$$\nu = \frac{1}{h}\frac{m_0c^2}{\sqrt{1-\beta^2}}. \tag{1.1.7}$$

These two frequencies, ν_1 and ν, are fundamentally different, in that the factor $\sqrt{1-\beta^2}$ enters into them differently. This is a difficulty that has intrigued me for a long time. It has brought me to the following conception, which I denote 'the theorem of phase harmony:'

"*A periodic phenomenon is seen by a stationary observer to exhibit the frequency* $\nu_1 = h^{-1}m_0c^2\sqrt{1-\beta^2}$ *that appears constantly in phase with a wave having frequency* $\nu = h^{-1}m_0c^2/\sqrt{1-\beta^2}$ *propagating in the same direction with velocity* $V = c/\beta$."

The proof is simple. Suppose that at $t = 0$ the phenomenon and wave have phase harmony. At time t then, the moving object has covered a distance equal to $x = \beta ct$ for which the phase equals $\nu_1 t = h^{-1}m_0c^2\sqrt{1-\beta^2}(x/\beta c)$. Likewise, the phase of the wave traversing the same distance is

$$\nu\left(t - \frac{\beta x}{c}\right) = \frac{m_0c^2}{h}\frac{1}{\sqrt{1-\beta^2}}\left(\frac{x}{\beta c} - \frac{\beta x}{c}\right) = \frac{m_0c^2}{h}\sqrt{1-\beta^2}\,\frac{x}{\beta c}. \tag{1.1.8}$$

As stated, we see here that phase harmony persists.

Additionally this theorem can be proved, essentially in the same way, but perhaps with greater impact, as follows. If t_0 is time for an observer at rest with respect to a moving body, i.e., his proper time, then the Lorentz transformation gives:

$$t_0 = \frac{1}{\sqrt{1-\beta^2}}\left(t - \frac{\beta x}{c}\right). \tag{1.1.9}$$

The periodic phenomenon we imagine is for this observer a sinusoidal function of $\nu_0 t_0$. For an observer at rest, this is the same sinusoid of $\nu_0\,(t - \beta x/c)\,/\sqrt{1-\beta^2}$ which represents a wave of frequency $\nu_0/\sqrt{1-\beta^2}$ propagating with velocity c/β in the direction of motion.

Here we must focus on the nature of the wave we imagine to exist. The fact that its velocity $V = c/\beta$ is necessarily greater than the velocity of light c, (β is always less that 1, except when mass is infinite or imaginary), shows that it can not represent transport of energy. Our theorem teaches us, moreover, that this wave represents a spacial distribution of *phase*, that is to say, it is a "*phase wave.*"

To make the last point more precise, consider a mechanical comparison, perhaps a bit crude, but that speaks to one's imagination. Consider a large, horizontal circular disk, from which identical weights are suspended on springs. Let the number of such systems per unit area, i.e., their density, diminish rapidly as one moves out from the centre of the disk, so that there is a high concentration at the centre. All the weights on springs have the same period; let us set them in motion with identical amplitudes and phases. The surface passing through the centre of gravity of the weights would be a plane oscillating up and down. This ensemble of systems is a crude analogue to a parcel of energy as we imagine it to be.

The description we have given conforms to that of an observer at rest with the disk. Were another observer moving uniformly with velocity $v = \beta c$ with respect to the disk to observe it, each weight for him appears to be a clock exhibiting Einstein retardation; further, the disk with its distribution of weights on springs, no longer is isotropic about the centre by cause of Lorentz contraction. But the central point here (in §1.3 it will be made more comprehensible), is that there is a dephasing of the motion of the weights. If, at a given moment in time a fixed observer considers the geometric location of the centre of mass of the various weights, he gets a cylindrical surface in a horizontal direction for which vertical slices parallel to the motion of the disk are sinusoids. This surface corresponds, in the case we envision, to our phase wave, for which, in accord with our general theorem, there is a surface moving with velocity c/β parallel to the disk and having a frequency of vibration on the fixed abscissa equal to that of a proper oscillation of a spring multiplied by $1/\sqrt{1-\beta^2}$. One sees finally with this example (which is our reason to pursue it) why a phase wave transports 'phase', but not energy.

The preceeding results seem to us to be very important, because with aid of the quantum hypothesis itself, they establish a link between motion of a material body and propagation of a wave, and thereby permit envisioning the possibility of a synthesis of these antagonistic theories on the nature of radiation. So, we note that a rectilinear phase wave is congruent with rectilinear motion of the body; and, FERMAT's principle applied to the wave specifies a ray, whereas MAUPERTUIS' principle applied to the material body specifies a rectilinear trajectory, which is in fact a ray for the wave. In CHAPTER 2, we shall generalise this coincidence.

1.2. Phase and Group Velocities

We must now explicate an important relationship existing between the velocity of a body in motion and a phase wave. If waves of nearby frequencies propagate in the same direction Ox with velocity V, which we call a phase velocity, these waves exhibit, by cause of superposition, a beat if the velocity V varies with the frequency ν. This phenomenon was studied by Lord RAYLEIGH specifically for the case of dispersive media.

Imagine two waves of nearby frequencies, ν and $\nu' = \nu + \delta\nu$, and velocities V and $V' = V + (dV/d\nu)\delta\nu$; their superposition leads analytically, while neglecting terms second order in $\delta\nu$ with respect to ν, to the following equation:

$$\sin(2\pi(\nu t - \frac{\nu x}{V} + \varphi)) + \sin(2\pi(\upsilon' t - \frac{\nu' x}{V'} + \varphi')) =$$

$$2\sin(2\pi(\nu t - \frac{\nu x}{V} + \psi))\cos(2\pi(\frac{\delta\nu}{2}t - x\frac{d\left(\frac{\nu}{V}\right)}{d\nu}\frac{\delta\nu}{2} + \psi')). \tag{1.2.1}$$

Thus we get a sinusoid for which the amplitude is modulated at frequency $\delta\nu$, because the sign of the cosine has little effect. This is a well known result. If one denotes with U the velocity of propagation of the beat, or group velocity, one finds:

$$\frac{1}{U} = \frac{d\left(\frac{\nu}{V}\right)}{d\nu}. \tag{1.2.2}$$

We return to phase waves. If one attributes a velocity $v = \beta c$ to the body, this does not fully determine the value of β, it only restricts the velocity to being betweenβ and $\beta + \delta\beta$; corresponding frequencies then span the interval $(\nu, \nu + \delta\nu)$.

We shall now prove a theorem that will be ultimately very useful: *The group velocity of phase waves equals the velocity of its associated body.* In effect this group velocity is determined by the above formula in which V and ν can be considered as functions of β because:

$$V = \frac{c}{\beta}, \quad \nu = \frac{1}{h}\frac{m_0 c^2}{\sqrt{1-\beta^2}}. \tag{1.2.3}$$

One may write:

$$U = \frac{\frac{d\nu}{d\beta}}{\frac{d\left(\frac{\nu}{V}\right)}{d\beta}}, \tag{1.2.4}$$

where

$$\frac{d\nu}{d\beta} = \frac{m_0 c^2}{h} \cdot \frac{\beta}{(1-\beta^2)^{3/2}},$$

$$d\frac{\left(\frac{\nu}{V}\right)}{d\beta} = \frac{m_0 c^2}{h} \cdot \frac{d\left(\frac{\beta}{\sqrt{1-\beta^2}}\right)}{d\beta} = \frac{m_0 c^2}{h}\frac{1}{(1-\beta^2)^{3/2}}; \tag{1.2.5}$$

so that:

(1.2.6) $$U = \beta c = v.$$

The phase wave group velocity is then actually equal to the body's velocity. This leads us to remark: in the wave theory of dispersion, except for absorption zones, velocity of energy transport equals group velocity[2]. Here, despite a different point of view, we get an analogous result, in so far as the velocity of a body is actually the velocity of energy displacement.

1.3. Phase waves in space-time

MINKOWSKI appears to have been first to obtain a simple geometric representation of the relationships introduced by EINSTEIN between space and time consisting of a Euclidian 4-dimensional space-time. To do so he took a Euclidean 3-space and added a fourth orthogonal dimension, namely time multiplied by $c\sqrt{-1}$. Nowadays one considers the fourth axis to be a real quantity ct, of a pseudo Euclidean, hyperbolic space for which the the fundamental invariant is $c^2dt^2 - dx^2 - dy^2 - dz^2$.

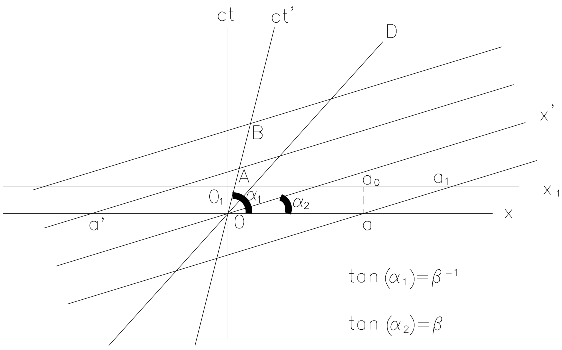

FIGURE 1.3.1. A Minkowski diagram showing worldlines for a body moving with velocity $v = \beta c$, (primed axis). OD is the light cone. Lines parallel to ox' are "lines of equal phase."

Let us consider now space-time for a stationary observer referred to four rectangular axes. Let x be in the direction of motion of a body on a chart together with the time axis and the above mentioned trajectory. (See Fig.: 1.3.1) Given these assumptions, the trajectory of the body will be a line inclined at an angle less than 45° to the time axis;

[2]See, for example: LÉON BRILLOUIN, *La Théorie des quanta et l'atom de Bohr*, CHAPTER 1.

this line is also the time axis for an observer at rest with respect to the body. Without loss of generality, let these two time axes pass through the origin.

If the velocity for a stationary observer of the moving body is βc, the slope of ot' has the value $1/\beta$. The line ox', i.e., the spacial axis of a frame at rest with respect to the body and passing through the origin, lies as the symmetrical reflection across the bisector of xot; this is easily shown analytically using Lorentz transformations, and shows directly that the limiting velocity of energy, c, is the same for all frames of reference. The slope of ox' is, therefore, β. If the comoving space of a moving body is the scene of an oscillating phenomenon, then the state of a comoving observer returns to the same place whenever time satisfies: $\overline{oA}/c = \overline{AB}/c$, which equals the proper time period, $T_0 = 1/\nu_0 = h/m_0c^2$, of the periodic phenomenon.

Lines parallel to ox' are, therefore, lines of equal 'phase' for the observer at rest with the body. The points $...a'$, o, $a...$ represent projections onto the space of an observer at rest with respect to the stationary frame at the instant 0; these two dimensional spaces in three dimensional space are planar two dimensional surfaces because all spaces under consideration here are Euclidean. When time progresses for a stationary observer, that section of space-time which for him is space, represented by a line parallel to ox, is displaced via uniform movement towards increasing t. One easily sees that planes of equal phase $...a'$, o, $a...$ are displaced in the space of a stationary observer with a velocity c/β. In effect, if the line ox_1 in Figure 1 represents the space of the observer fixed at $t = 1$, for him $\overline{aa_0} = c$. The phase that for $t = 0$ one finds at a , is now found at a_1; for the stationary observer, it is therefore displaced in his space by the distance a_0a_1 in the direction ox by a unit of time. One may say therefore that its velocity is:

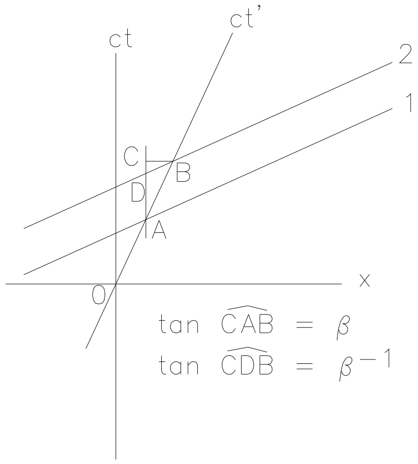

FIGURE 1.3.2. A Minkowski diagram: details, showing the trigonometric relationships yielding the frequency.

$$V = a_0a_1 = aa_0 \coth(\angle x0x') = \frac{c}{\beta}. \qquad (1.3.1)$$

The ensemble of equal phase planes constitutes what we have denoted a 'phase wave.'

To determine the frequency, refer to Fig. 1.3.2.

Lines 1 and 2 represent two successive equal phase planes of a stationary observer. $\overline{AB}$ is, as we said, equal to c times the proper period $T_0 = h/m_0c^2$.

AC, the projection of AB on the axis Ot, is equal to:

$$cT_1 = cT_0 \frac{1}{\sqrt{1-\beta^2}}. \tag{1.3.2}$$

This result is a simple application of trigonometry; whenever, we emphasise, trigonometry is used on the plane xot, it is vitally necessary to keep in mind that there is a peculiar anisotropism of this plane. The triangle ABC yields:

$$\begin{aligned} \overline{(AB)^2} = \overline{(AC)^2} - \overline{(CB)^2} &= \overline{(AC)^2}(1-\tan^2(\angle CAB)), \\ &= \overline{(AC)^2}(1-\beta^2), \\ \overline{AC} &= \frac{\overline{AB}}{\sqrt{1-\beta^2}}, \quad \text{q.e.d.} \end{aligned} \tag{1.3.3}$$

The frequency $1/T_1$is that which the periodic phenomenon appears to have for a stationary observer using his eyes from his position. That is:

$$\nu_1 = \nu_0\sqrt{1-\beta^2} = \frac{m_0c^2}{h}\sqrt{1-\beta^2}. \tag{1.3.4}$$

The period of these waves at a point in space for a stationary observer is given not by $\overline{AC}/c$, but by $\overline{AD}/c$. Let us calculate it.

For the small triangle BCD, one finds that:

$$\frac{\overline{CB}}{\overline{DC}} = \frac{1}{\beta}, \quad \text{or} \quad \overline{DC} = \beta\overline{CB} = \beta^2\overline{AC}. \tag{1.3.5}$$

But, in so far as $\overline{AD} = \overline{AC} - \overline{DC} = \overline{AC}(1-\beta^2)$, the new period equals:

$$T = \frac{1}{c}\overline{AC}(1-\beta^2) = T_0\sqrt{1-\beta^2}, \tag{1.3.6}$$

and the frequency ν of this wave is given by:

$$\nu = \frac{1}{T} = \frac{\nu_0}{\sqrt{1-\beta^2}} = \frac{m_0c^2}{h\sqrt{1-\beta^2}}. \tag{1.3.7}$$

Thus we obtain again all the results obtained analytically in §1.1, but now we see better how it relates to general concepts of space-time and why dephasing of periodic movements takes place differently depending on the definition of simultaneity in relativity.

CHAPTER 2

The principles of MAUPERTUIS and FERMAT

2.1. Motivation

We wish to extend the results of CHAPTER 1 to the case in which motion is no longer rectilinear and uniform. Variable motion presupposes a force field acting on a body. As far as we know there are only two types of fields: electromagnetic and gravitational. The General Theory of Relativity attributes gravitational force to curved space-time. In this work we shall leave all considerations on gravity aside, and return to them elsewhere. Thus, for present purposes, a field is an electromagnetic field and our study is on its affects on the motion of a charged particle.

We must expect to encounter significant difficulties in this chapter in so far as Relativity, a sure guide for uniform motion, is just as unsure for nonuniform motion. During a recent visit of M. EINSTEIN to Paris, M. PAINLEVÉ raised several interesting objections to Relativity; M. LANGEVIN was able to deflect them easily because each involved acceleration, for which Lorentz-Einstein transformations don't pertain, even not to uniform motion. Such arguments by illustrious mathematicians have thereby shown again that application of EINSTEIN's ideas is very problematical whenever there is acceleration involved; and in this sense are very instructive. The methods used in CHAPTER 1 can not help us here.

The phase wave that accompanies a body, if it is always to comply with our notions, has properties that depend on the nature of the body, since its frequency, for example, is determined by its total energy. It seems natural, therefore, to suppose that, if a force field affects particle motion, it also must have some affect on propagation of phase waves. Guided by the idea of a fundamental identity of the principle of least action and Fermat's principle, I have conducted my researches from the start by supposing that given the total energy of a body, and therefore the frequency of its phase wave, trajectories of one are rays of the other. This has lead me to a very satisfying result which shall be delineated in CHAPTER 3 in light of BOHR's interatomic stability conditions. Unfortunately, it needs hypothetical inputs on the value of the propagation velocity, V , of the phase wave at each point of the field that are rather arbitrary. We shall therfore make use of another method that seems to us more general and satisfactory. We shall study on the one hand the relativistic version of the mechanical principle of least action in its Hamiltonian and Maupertuisian form, and on the other hand from a very general point of view, the propagation of waves according to FERMAT. We shall then propose a synthesis of these two, which, perhaps, can be disputed, but which has incontestable elegance. Moreover, we shall find a solution to the problem we have posed.

2.2. Two principles of least action in classical dynamics

In classical dynamics, the principle of least action is introduced as follows:

The equations of dynamics can be deduced from the fact that the integral $\int_{t_1}^{t_2} \mathcal{L}\, dt$, between fixed time limits, t_1 and t_2, and specified by parameters q_i which give the state of the system, *has a stationary value.* By definition, $\mathcal{L}$, known as LAGRANGE's function, or Lagrangian, depends on q_i and $\dot{q}_i = dq_i/dt$.

Thus, one has:

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0. \tag{2.2.1}$$

From this one deduces the equations of motion using the calculus of variations given by LAGRANGE:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}. \tag{2.2.2}$$

where there are as many equations as there are q_i.

It remains now only to define $\mathcal{L}$. Classical dynamics calls for:

$$\mathcal{L} = E_{\text{kin.}} - E_{\text{pot.}}, \tag{2.2.3}$$

i.e., the difference in kinetic and potential energy. We shall see below that relativistic dynamics uses a different form for $\mathcal{L}$.

Let us now proceed to the principle of least action of MAUPERTUIS. To begin, we note that LAGRANGE's equations in the general form given above, admit a first integral called the "system energy" which equals:

$$W = -\mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \tag{2.2.4}$$

under the condition that the function $\mathcal{L}$ does not depend explicitely on time, which we shall take to be the case below.

$$\begin{aligned} \frac{dW}{dt} &= \sum_i \left(-\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i \right) \\ &= \sum_i \dot{q}_i \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right], \end{aligned} \tag{2.2.5}$$

which according to LAGRANGE, is null. Therefore:

$$W = \text{const.} \tag{2.2.6}$$

We now apply HAMILTON's principle to all "variable" trajectories constrained to initial position a and final position b for which energy is a constant. One may write, as W , t_1 and t_2 are all constant:

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \delta \int_{t_1}^{t_2} (\mathcal{L} + W) dt = 0, \tag{2.2.7}$$

or else:

$$\delta \int_{t_1}^{t_2} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i dt = \delta \int_A^B \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} dq_i = 0, \tag{2.2.8}$$

the last integral is intended for evaluation over all values of q_i definitely contained between states A and B of the sort for which time does not enter; there is, therefore, no further place here in this new form to impose any time constraints. On the contrary, all varied trajectories correspond to the same value of energy, W.[1]

In the following we use classical canonical equations: $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. MAUPERTUIS' principle may be now be written:

$$\delta \int_A^B \sum_i p_i dq_i = 0, \tag{2.2.9}$$

in classical dynamics where $\mathcal{L} = E_{\text{kin.}} - E_{\text{pot.}}$ is independent of $\dot{q}_i$ and $E_{\text{kin.}}$ is a homogeneous quadratic function. By virtue of EULER's Theorem, the following holds:

$$\sum_i p_i dq_i = \sum_i p_i \dot{q}_i dt = 2E_{\text{kin}dt.}. \tag{2.2.10}$$

For a material point body, $E_{\text{kin.}} = mv^2/2$ and the principle of least action takes its oldest known form:

$$\delta \int_A^B mvdl = 0. \tag{2.2.11}$$

where dl, is a differential element of a trajectory.

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[1] *Footnote added to the German translation*: To make this proof rigorous, it is necessary, as is well known, to also vary t_1 and t_2; but, because of the time independence of the result, our argument is not false.

2.3. The two principles of least action for electron dynamics

We turn now to the matter of relativistic dynamics for an electron. Here by 'electron' we mean simply a massive particle with charge. We take it that an electron outside any field posses a proper mass m_e; and carries charge e.

We now return to space-time, where space coordinates are labelled x^1, x^2 and x^3, the coordinate ct is denoted by x^4. The invariant fundamental differential of length is defined by:

$$ds = \sqrt{(dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}. \tag{2.3.1}$$

In this section and below we shall employ certain tensor expressions.

A world line has at each point a tangent defined by a vector, "world-velocity" of unit length whose contravariant components are given by:

$$u^i = \frac{dx^i}{ds}, \quad (i = 1, 2, 3, 4). \tag{2.3.2}$$

One sees immediately that $u^i u_i = 1$.

Let a moving body prescribe a world line; when it passes a particular point, it has a velocity $v = \beta c$ with components v_x, v_y, v_z. The components of its world-velocity are:

$$u_1 = -u^1 = -\frac{v_x}{c\sqrt{1-\beta^2}}, \quad u_2 = -u^2 = -\frac{v_y}{c\sqrt{1-\beta^2}}$$

$$u_3 = -u^3 = -\frac{v_z}{c\sqrt{1-\beta^2}}, \quad u_4 = u^4 = \frac{1}{c\sqrt{1-\beta^2}}. \tag{2.3.3}$$

To define an electromagnetic field, we introduce another world-vector whose components express the vector potential $\vec{a}$ and scalar potential Ψ by the relations:

$$\varphi_1 = -\varphi^1 = -a_x; \quad \varphi_2 = -\varphi^2 = -a_y;$$

$$\varphi_3 = -\varphi^3 = -a_z; \quad \varphi_4 = \varphi^4 = \frac{1}{c}\Psi. \tag{2.3.4}$$

We consider now two points P and Q in space-time corresponding to two given values of the coordinates of space-time. We imagine an integral taken along a curvilinear world line from P to Q; naturally the function to be integrated must be invariant.

Let:

$$\int_P^Q (-m_0 c - e\varphi_i u^i) ds = \int_P^Q (-m_0 c u_i - e\varphi_i) u^i ds, \tag{2.3.5}$$

be this integral. HAMILTON's Principle affirms that if a world-line goes from P to Q, it has a form which give this integral a stationary value.

Let us define a third world-vector by the relations:

$$J_i = m_0 c u_i + e\varphi_i, \quad (i = 1, 2, 3, 4), \tag{2.3.6}$$

the statement of least action then gives:

$$\delta \int_P^Q J_i dx^i = 0. \tag{2.3.7}$$

Below we shall give a physical interpretation to the world vector J.

Now let us return to the usual form of dynamics equations in that we replace in the first equation for the action, ds by $cdt\sqrt{1-\beta^2}$. Thus, we obtain:

$$\delta \int_{t_1}^{t_2} [-m_0 c^2 \sqrt{1-\beta^2} - ec\varphi_4 - e(\vec{\varphi} \cdot \vec{v})]dt = 0, \tag{2.3.8}$$

where t_1 and t_2 correspond to points P and Q in space-time.

If there is a purely electrostatic field, then $\vec{\varphi}$ is zero and the Lagrangian takes the simple form:

$$\mathcal{L} = -m_0 c^2 \sqrt{1-\beta^2} - e\Psi. \tag{2.3.9}$$

In any case, HAMILTON's Principle always has the form $\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0$, it always leads to LAGRANGE's equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}, \quad (i = 1, 2, 3). \tag{2.3.10}$$

In each case for which potentials do not depend on time, conservation of energy obtains:

$$W = -\mathcal{L} + \sum_i p_i dq_i = \text{const.}, \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad (i = 1, 2, 3). \tag{2.3.11}$$

Following exactly the same argument as above, one also can obtain MAUPERTUIS' Principle:

$$\delta \int_A^B \sum p_i dq_i = 0, \tag{2.3.12}$$

where A and B are the two points in space corresponding to said points P and Q in space-time.

The quantities p_i, equal to partial derivatives of $\mathcal{L}$ with respect to velocities $\dot{q}_i$, define the "momentum" vector: $\vec{p}$. If there is no magnetic field (irrespective of whether there is an electric field) , $\vec{p}$ equals:

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1-\beta^2}}. \tag{2.3.13}$$

It is therefore identical to momentum and MAUPERTUIS' integral of action takes just the simple form proposed by MAUPERTUIS himself, with the difference that mass is now variable according to Lorentz transformations.

If there is also a magnetic field, one finds that the components of momentum take the form:

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1-\beta^2}} + e\vec{a}. \tag{2.3.14}$$

In this case there no longer is an identity between $\vec{p}$ and momentum; therefore an expression of the integral of motion is more complicated.

Consider a moving body in a field for which total energy is given; at every point of the given field which a body can sample, its velocity is specified by conservation of energy, whilst *a priori* its direction may vary. The form of the expression of $\vec{p}$ in an electrostatic field reveals that vector momentum has the same magnitude regardless of its direction. This is not the case if there is a magnetic field; the magnitude of $\vec{p}$ depends on the angle between the chosen direction and the vector potential as can be seen in its effect on $\vec{p} \cdot \vec{p}$. We shall make use of this fact below.

Finally, let us return to the issue of the physical interpretation of a world-vector $\vec{\mathbf{J}}$ from which a Hamiltonian depends. We have defined it as**:**

$$\vec{\mathbf{J}} = m_0 c \vec{\mathbf{u}} + e\vec{\varphi}. \tag{2.3.15}$$

Expanding $\vec{u}$ and $\vec{\varphi}$, one finds:

$$\vec{J} = -\vec{p}, \quad J_4 = \frac{W}{c}. \tag{2.3.16}$$

Thus, we have constructed the renowned "world momentum" which unifies energy and momentum.

From:

$$\delta \int_P^Q J_i dx^i = 0, \quad (i = 1, 2, 3, 4), \tag{2.3.17}$$

one can simplify, if J_4 is constant, to:

$$\delta \int_A^B J_i dx^i = 0, \quad (i = 1, 2, 3). \tag{2.3.18}$$

This is the most direct manner to go from one version of least action to the other.

2.4. Wave propagation; FERMAT's Principle

We shall study now phase wave propagation using a method parallel to that of the last two sections. To do so, we take a very general and broad viewpoint on space-time.

Consider the function $\sin\varphi$ in which a differential of φ is taken to depend on space-time coordinates x_i. There are an infinity of lines in space-time along which a function of φ is constant.

The theory of undulations, especially as promulgated by HUYGENS and FRESNEL, leads us to distinguish among these lines certain of them that are projections onto the space of an observer, which are there "rays" in the optical sense.

Let two points such as those above, P and Q, be two points in space-time. If a world ray passes through these two points, what law determines its form?

Consider the line integral $\int_P^Q d\varphi$, let us suppose that a law equivalent to HAMILTON's but now for world rays takes the form:

$$\delta\int_P^Q d\varphi = 0. \tag{2.4.1}$$

This integral should be, in fact, stationary; otherwise, perturbations breaking phase concordance after a given crossing point, would propagate forward to make the phase then be discordant at a second crossing.

The phase φ is an invariant, so we may posit:

$$d\varphi = 2\pi\sum_i O_i x^i, \tag{2.4.2}$$

where O_i , usually functions of x^i, constitute a world vector, the world wave. If l is the direction of a ray in the usual sense, it is the custom to envision for $d\varphi$ the form:

$$d\varphi = 2\pi(\nu dt - \frac{\nu}{V}dl), \tag{2.4.3}$$

where ν is the frequency and V is the velocity of propagation. Thus, one may write:

$$O_i = -\frac{\nu}{V}\cos(x_i, t), \quad O_4 = \frac{\nu}{V}. \tag{2.4.4}$$

The world wave vector can be decomposed, therefore, into a component proportional to frequency and a space vector $\vec{n}$ aimed in the direction of propagation and having a magnitude ν/V. We shall call this vector a "wave number", as it is proportional to the inverse of wave length. If the frequency ν is constant, we are then lead to the Hamiltonian:

$$\delta\int_P^Q O_i dx^i = 0, \tag{2.4.5}$$

in the Maupertuisian form:

$$\delta\int_A^B \sum_i O_i dx^i = 0, \tag{2.4.6}$$

where A and B are points in space corresponding to P and Q .

By substituting for $\vec{O}$ its values, one gets:

$$\delta\int_A^B \frac{\nu dl}{V} = 0. \tag{2.4.7}$$

This statement of MAUPERTUIS' Principle constitutes FERMAT's Principle also. Just as in §2.3, in order to find the trajectory of a moving body of given total energy, it suffices to know the distribution of the vector field $\vec{p}$, the same is true to find the ray passing through two points, it suffices to know the wave vector field which determines at each point and for each direction, the velocity of propagation.

2.5. Extending the quantum relation

Thus, we have reached the final stage of this chapter. At the start we posed the question: when a body moves in a force field, how does its phase wave propagate? Instead of searching by trial and error, as I did in the beginning, to determine the velocity of propagation at each point for each direction, I shall extend the quantum relation, a bit hypothetically perhaps. but in full accord with the spirit of Relativity.

We are constantly drawn to writing $h\nu = w$, where w is the total energy of the body and ν is the frequency of its phase wave. On the other hand, in the preceeding sections we defined two world vectors J and O which play symmetric roles in the study of motion of bodies and waves.

In light of these vectors, the relation $h\nu = w$ can be written:

$$O_4 = \frac{1}{h} J_4. \tag{2.5.1}$$

However, the fact that two vectors have one equal component, does not prove that the other components are equal. Nevertheless, by virtue of an obvious generalisation, we pose that:

$$O_i = \frac{1}{h} J_i, \quad (1, 2, 3, 4). \tag{2.5.2}$$

The variation $d\varphi$ relative to an infinitesimally small portion of the phase wave has the value:

$$d\varphi = 2\pi O_i dx^i = \frac{2\pi}{h} J_i dx^i. \tag{2.5.3}$$

FERMAT 's Principle becomes then:

$$\delta \int_A^B \sum_i^3 J_i dx^i = \delta \int_A^B \sum_i^3 p_i dx^i = 0. \tag{2.5.4}$$

Thus, we get the following statement:

Fermat's Principle applied to a phase wave is equivalent to Maupertuis' Principle applied to a particle in motion; the possible trajectories of the particle are identical to the rays of the phase wave.

We believe that the idea of an equivalence between the two great principles of Geometric Optics and Dynamics might be a precise guide for effecting the synthesis of waves and quanta.

The hypothetical proportionality of J and O is a sort of extention of the quantum relation, which in its original form is manifestly insufficient because it involves energy but not its inseparable partner: momentum. This new statement is much more satisfying since it is expressed as the equality of two world vectors.

2.6. Examples and discussion

The general notions in the last section need to be applied to particular cases for the purpose of explicating their exact meaning.

a) Let us consider first linear motion of a free particle. The hypotheses from CHAPTER 1 with the help of Special Relativity allow us to handle this case. We wish to check if the predicted propagation velocity for phase waves:

$$V = \frac{c}{\beta}, \tag{2.6.1}$$

comes back out of the formalism.

Here we must take:

$$\nu = \frac{W}{h} = \frac{m_0 c^2}{h\sqrt{1-\beta^2}}, \tag{2.6.2}$$

$$\frac{1}{h}\sum_1^3 p_i dq_i = \frac{1}{h}\frac{m_0 \beta^2 c^2}{\sqrt{1-\beta^2}} dt = \frac{1}{h}\frac{m_0 \beta c}{\sqrt{1-\beta^2}} dl = \frac{\nu dl}{V}, \tag{2.6.3}$$

from which we get: $V = c/\beta$. Moreover, we have given it an interpretation from a space-time perspective.

b) Consider an electron in an electric field (Bohr atom). The frequency of the phase wave can be taken to be energy divided by h, where energy is given by:

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} + e\psi = h\nu. \tag{2.6.4}$$

When there is no magnetic field, one has simply:

$$p_x = \frac{m_0 v_x}{\sqrt{1-\beta^2}}, \quad \text{etc.}, \tag{2.6.5}$$

$$\frac{1}{h}\sum_1^3 p_i dq_i = \frac{1}{h}\frac{m_0\beta c}{\sqrt{1-\beta^2}}dl = \frac{\nu}{V}dl, \tag{2.6.6}$$

from which we get:

$$\begin{aligned} V &= \frac{\frac{m_0c^2}{\sqrt{1-\beta^2}} + e\psi}{\frac{m_0\beta c}{\sqrt{1-\beta^2}}} = \frac{c}{\beta}\left(1 + \frac{e\psi\sqrt{1-\beta^2}}{m_0c^2}\right), \\ &= \frac{c}{\beta}\left(1 + \frac{e\psi}{W - e\psi}\right) = \frac{c}{\beta}\frac{W}{W - e\psi}. \end{aligned} \tag{2.6.7}$$

This result requires some comment. From a physical point of view, this shows that, a phase wave with frequency $\nu = W/h$ propagates at each point with a different velocity depending on potential energy. The velocity V depends on ψ directly as given by $e\psi/(W - e\psi)$ (a quantity generally small with respect to 1) and indirectly on β, which at each point is to be calculated from W and ψ.

Further, it is to be noticed that V is a function of the mass and charge of the moving particle. This may seem strange; however, it is less unreal that it appears. Consider an electron whose centre moves with velocity v; which, according to classical notions, is located at point P, expressed in a coordinate system fixed to the particle, and to which there is associated electromagnetic energy. We assume that after traversing the region R in Fig. (2.6.1), with its more or less complicated electromagnetic field, the particle has the same speed but new direction.

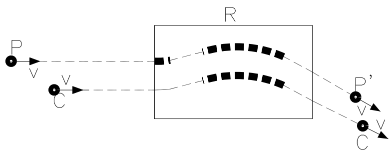

FIGURE 2.6.1. Electron energy-transport through a region with fields.

The point P is then transfered to point P', and one can say that the starting energy at P was transported to point P'. The transfer of this energy through region R, even knowing the fields therein in detail, only can be specified in terms of a charge and mass. This may seem bizarre in that we are accustomed to thinking that charge and mass (as well as momentum and energy) are properties vested in the centre of an electron. In connection with a phase wave, which in our conceptions is a substantial part of the electron, its propagation also must be given in terms of mass and charge.

Let us return now to the results from CHAPTER 1 for the case of uniform motion. We have been drawn into considering a phase wave as being due to the intersection of the space of the fixed observer with the past, present and future spaces of a comoving observer. We might be tempted here again to recover the value of V given above, by considering successive "phases" of the particle in motion and to determine displacement relative to a stationary observer by means of sections of his space as states of equal phase. Unfortunately, one encounters here three large difficulties. Contemporary Relativity does not instruct us how a non uniformly moving observer is at each moment to isolate his pure space from space-time; there does not appear to be good reason to assume that this separation is just the same as for uniform motion. But even were this difficulty overcome, there are still obstacles. A uniformly moving particle would be described by a comoving observer always in the same way; a conclusion that follows for uniform motion from equivalence of Galilean systems. Thus, if a uniformly moving particle with comoving observer is associated with a periodic phenomenon always having the same phase, then the same velocity will always pertain and therefore the methods in CHAPTER 1 are applicable. If motion is not uniform, however, a description by a comoving observer can no longer be the same, and we just don't know how associated periodic phenomenon would be described or whether to each point in space there corresponds the same phase.

Maybe, one might reverse this problem, and accept results obtained in this chapter by different methods in an attempt to find how to formulate relativistically the issue of variable motion, in order to achieve the same conclusions. We can not deal with this difficult problem.

c.) Consider the general case of a charge in an electromagnetic field, where:

$$h\nu = W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} + e\psi. \tag{2.6.8}$$

As we have shown above, in this case:

$$p_x = \frac{m_0 v_x}{\sqrt{1-\beta^2}} + e a_x, \quad \text{etc.,} \tag{2.6.9}$$

where a_x, a_y, a_z are components of the vector potential.

Thus,

$$\frac{1}{h}\sum_1^3 p_i dq_i = \frac{1}{h}\frac{m_0 \beta c}{\sqrt{1-\beta^2}} + \frac{e}{h} a_l dl = \frac{\nu dl}{V}. \tag{2.6.10}$$

So that one finds:

$$V = \frac{\frac{m_0 c^2}{\sqrt{1-\beta^2}} + e\psi}{\frac{m_0 \beta c}{\sqrt{1-\beta^2}} + e a_l} = \frac{c}{\beta}\frac{W}{W - e\psi}\frac{1}{1 + e\frac{a_l}{G}}. \tag{2.6.11}$$

where G is the momentum and a_l is the projection of the vector potential onto the direction l.

The environment at each point is no longer isotropic. The velocity V varies with the direction, and the particle's velocity $\vec{v}$ no longer has the same direction as the normal to the phase wave defined by $\vec{p} = h\vec{n}$. That the ray doesn't coincide with the wave normal is virtually the classical definition of anisotropic media.

One can question here the theorem on the equality of a particle's velocity $v = \beta c$ with the group velocity of its phase wave.

At the start, we note that the velocity of a phase wave is defined by:

$$\frac{1}{h}\sum_{1}^{3} p_i dq_i = \frac{1}{h}\sum_{1}^{3} p_i \frac{dq_i}{dl} dl = \frac{\nu}{V} dl, \tag{2.6.12}$$

where ν/V does not equal p/h because dl and p don't have the same direction.

We may, without loss of generality, take it that the x axis is parallel to the motion at the point where p_x is the projection of p onto this direction. One then has the definition:

$$\frac{\nu}{V} = \frac{p_x}{h}. \tag{2.6.13}$$

The first canonical equation then provides the relation:

$$\frac{dq_x}{dt} = v = \beta c = \frac{\partial W}{\partial p_x} = \frac{\partial(h\nu)}{\partial\left(\frac{h\nu}{V}\right)} = U, \tag{2.6.14}$$

where U is the group velocity following the ray.

The result from §1.2 is therefore fully general and the first group of HAMILTON's equations follows directly.

CHAPTER 3

Quantum stability conditions for trajectories

3.1. Bohr-Sommerfeld stability conditions

In atomic theory, M. BOHR was first to enunciate the idea that among the closed trajectories that an electron may assume about a positive charge centre, only certain ones are stable, the remaining are by nature transitory and may be ignored. If we focus on circular motion, then there is only one degree of freedom, and BOHR's Principle is given as follows: *Only those circular orbits are stable for which the action is a multiple of $h/2\pi$, where h is* PLANCK*'s constant*. That is:

$$m_0\omega R^2 = n\frac{h}{2\pi} \quad (n \text{ integer}), \tag{3.1.1}$$

or, alternately:

$$\int_0^{2\pi} p_\theta d\theta = nh, \tag{3.1.2}$$

where θ is a Lagrangian coordinate (i.e., q) and p_θ its canonical momentum.

MM. SOMMERFELD and WILSON, to extend this principle to the case of more degrees of freedom, have shown that it is generally possible to chose coordinates, q_i, for which the quantisation condition is:

$$\oint p_i dq_i = n_i h, \quad (n_i \text{ integer}), \tag{3.1.3}$$

where integration is over the whole domain of the coordinate.

In 1917, M. EINSTEIN gave this condition for quantisation an invariant form with respect to changes in coordinates[1]. For the case of *closed* orbits, it is as follows:

$$\oint \sum_1^3 p_i dq_i = nh, \quad (n \text{ integer}), \tag{3.1.4}$$

where it is to be valid along the total orbit. One herewith recognises MAUPERTUIS' integral of action to be important also for quantum theory. This integral does not depend

[1] EINSTEIN , A., Zum quantensatz von SOMMERFELD und EPSTEIN, *Ber. der deutschen Phys. Ges.* (1917) p. 82.

at all on a choice of space coordinates according to a property that expresses the covariant character of the vector components p_i of momentum. It is defined by the classical technique of JACOBI as a total integral of the particular differential equation:

$$H\left(\frac{\partial s}{\partial q_i}, q_i\right) = W; \quad i = 1, 2, ..., f, \tag{3.1.5}$$

where the total integral contains f arbitrary constants of integration of which one is energy, W. If there is only one degree of freedom, EINSTEIN's relation fixes the value of energy, W; if there are more than one (in the most important case, that of motion of an electron in an interatomic field, there are *a priori* three), one imposes a condition among W and the $n-1$ others; which would be the case for Keplerian ellipses were it not for relativistic variation of mass with velocity. However, if motion is quasi-periodic, which, moreover, always is the case for the above variation, it is possible to find coordinates that oscillate between its limit values (i.e., librations), and there is an infinity of pseudo-periods approximately equal to whole multiples of libration periods. At the end of each pseudo-period, the particle returns to a state very near its initial state. EINSTEIN's equation applied to each of these pseudo-periods leads to an infinity of conditions which are compatible only if the many conditions of SOMMERFELD are met; in which case all constants are determined, there is no longer indeterminism.

JACOBI's equation, angular variables and the residue theorem serve well to determine SOMMERFELD's integrals. This matter has been the subject of numerous books in recent years and is summarised in SOMMERFELD's beautiful book: *Atombau und Spectrallinien* (édition fran caise, traduction BELLENOT, BLANCHARD éditeur, 1923). We shall not pursue that here, but limit ourselves to remarking that the quantisation problem resides entirely on EINSTEIN's condition for closed orbits. If one succeeds in interpreting this condition, then with the same stroke one clarifies the question of stable trajectories.

3.2. The interpretation of EINSTEIN's condition

The phase wave concept permits explanation of EINSTEIN's condition. One result from CHAPTER 2 is that a trajectory of a moving particle is identical to a ray of a phase wave, along which frequency is constant (because total energy is constant) and with variable velocity, whose value we shall not attempt to calculate. Propagation is, therefore, analogue to a liquid wave in a channel, closed on itself but of variable depth. It is physically obvious, that to have a stable regime, the length of the channel must be resonant with the wave; in other words, the points of a wave located at whole multiples of the wave length l, must be in phase. The resonance condition is $l = n\lambda$ if the wave length is constant, and $\oint(\nu/V)dl = n$ (integer) in the general case.

The integral involved here is that from FERMAT's Principle; or, as we have shown, MAUPERTUIS' integral of action divided by h. Thus, the resonance condition can be identified with the stability condition from quantum theory.

This beautiful result, for which the demonstration is immediate if one admits the notions from the previous chapter, constitutes the best justification that we can give for our attack on the problem of interpreting quanta.

In the particular case of closed circular Bohr orbits in an atom, one gets: $m_0 \oint v dl = 2\pi R m_0 v = nh$ where $v = R\omega$ when ω is angular velocity,

$$m_0 \omega R^2 = n \frac{h}{2\pi}. \tag{3.2.1}$$

This is exactly BOHR's fundamental formula.

From this we see why certain orbits are stable; but, we have ignored passage from one to another stable orbit. A theory for such a transition can't be studied without a modified version of electrodynamics, which so far we do not have.

3.3. SOMMERFELD's conditions on quasiperiodic motion

I aim to show that if the stability condition for a closed orbit is $\oint \sum_1^3 p_i dq_i = nh$, then the stability condition for quasi-periodic motion is necessarily: $\oint p_i dq_i = n_i h$ (n_i integer $i = 1, 2, 3$). SOMMERFELD's multiple conditions bring us back again to phase wave resonance.

At the start we should note that an electron has finite dimensions; then, if, as we saw above, stability conditions depend on the interaction with its proper phase wave, there must be coherence with phase waves passing by at small distances, say on the order of its radius (10^{-13}cm.). If we don't admit this, then we must consider the electron as a pure point particle with a radius of zero, and this is not physically plausible.

Let us recall now a property of quasi-periodic trajectories. If M is the centre of a moving body at an instant along its trajectory, and if one considers a sphere of small but finite arbitrary radius R centred on M, it is possible to find an infinity of time intervals such that at the end of each, the body has returned to a point in a sphere of radius R. Moreover, each of these time intervals or "near periods" τ must satisfy:

$$\tau = n_1 T_1 + \varepsilon_1 = n_2 T_2 + \varepsilon_2 = n_3 T_3 + \varepsilon_3, \tag{3.3.1}$$

where T_i are the variable periods (librations) of the coordinates q_i. The quantities ε_i can always be rendered smaller than a fixed, small but finite interval: η. The shorter η is chosen to be, the longer the shortest of the τ will be.

Suppose that the radius R is chosen to be equal the maximum distance of action of the electron's phase wave, a distance defined above. Now, one may apply to each period approaching τ, the concordance condition for phase waves in the form:

$$\int_0^{\tau} \sum_1^3 p_i dq_i = nh, \tag{3.3.2}$$

where we may also write:

$$\sum_i (n_i \int_0^{T_i} p_i \dot{q}_i dt + \varepsilon_i (p_i \dot{q}_i)_\tau) = nh. \tag{3.3.3}$$

But a resonance condition is never rigorously satisfied. If a mathematician demands that for a resonance the difference be exactly $n \times 2\pi$, a physicist accepts $n \times 2\pi \pm \alpha$, where α is less than a small but finite quantity ε which may be considered the smallest physically sensible possibility.

The quantities p_i and q_i remain finite in the course of their evolution so that one may find six other quantities, P_i and Q_i, for which it is alway true that:

$$p_i < P_i; \quad \dot{q}_i < \dot{Q}_i, \quad (i = 1, 2, 3). \tag{3.3.4}$$

Choosing now the limit η such that $\eta \sum_1^3 P_i \dot{Q}_i < \varepsilon h/2\pi$; we see that, it does not matter what the quasi period is, which permits neglecting the terms ε_i to write:

$$\sum_{i=1}^{3} n_i \int_0^{T_i} p_i \dot{q}_i = nh. \tag{3.3.5}$$

On the left side, n_i are known whole numbers, while on the right n is an arbitrary whole number. We have thus an infinity of similar equations with different values of n_i. To satisfy them it is necessary and sufficient that each of the integrals:

$$\int_0^{T_i} p_i \dot{q}_i dt = \oint p_i dq_i, \tag{3.3.6}$$

equals an integer number times h.

These are actually SOMMERFELD's conditions.

The preceeding demonstration appears to be rigorous. However, there is an objection that should be rebutted. Stability conditions don't play a role for times shorter than τ; if waiting times of millions of years are involved, one could say they never play a role. This objection is not well founded, however, because the periods τ are very large with respect to the librations T_i, but may be very small with respect to our scale of time measurements; in an atom, the periods T_i are in effect, on the order of 10^{-15} to 10^{-20} seconds.

One can estimate the limit of the periods in the case of the L_2 trajectory for hydrogen from SOMMERFELD. Rotation of the perihelion during one libration period of a radius vector is on the order of $2\pi \times 10^{-5}$. The shortest periods then are about 10^5 times the period of the radial vector (10^{-15} seconds), or about 10^{-10} seconds. Thus, it seems that stability conditions come into play in time intervals inaccessible to our experience of time, and, therefore, that trajectories "without resonances" can easily be taken not to exist on a practical scale.

The principles delineated above were borrowed from M. BRILLOUIN who wrote in his thesis (p. 351): "The reason that MAUPERTUIS' integral equals an integer time *h*, is that each integral is relative to each variable and, over a period, takes a whole number of quanta; This is the reason SOMMERFELD posited his quantum conditions."

CHAPTER 4

Motion quantisation for two charges

4.1. Particular difficulties

In the preceeding chapters we repeatedly envisioned an "isolated parcel" of energy. This notion is clear when it pertains to a charged particle (proton or electron, say) well removed from other charged bodies. But if the charge centres interact, this notion is no longer so clear. There is here a difficulty that is not really a part of the subject of this work and is not elucidated by current relativistic dynamics.

To better understand this difficulty, consider a proton (hydrogen ion) of proper mass M_0 and an electron of proper mass m_0. If these two are far removed one from another, then their interaction is negligible, and one can apply easily the principle of inertia of energy: a proton has internal energy M_0c^2, whilst an electron has m_0c^2. Total internal energy is therefore: $(M_0+m_0)c^2$. But if the two are close to each other, with mutual potential energy $-P(<0)$, how must it be taken into account? Evidently it would be: $(M_0+m_0)c^2-P$, so should we consider that a proton always has mass M_0 and an electron m_0? Should not potential energy be parcelled between these two components of this system by attributing to an electron a proper mass $m_0-\alpha P/c^2$, and to a proton: $M_0-(1-\alpha)P/c^2$? In which case, what is the value of α and does it depend on M_0 or m_0?

In BOHR's and SOMMERFELD's atomic theories, one takes it that an electron always has proper mass m_0 at its position in the electrostatic field of a proton. Potential energy is always much less than internal energy m_0c^2, a hypothesis that is not inexact, but nothing says that it is fully rigorous. One can easily calculate the order of magnitude of the largest correction (corresponding to $\alpha=1$) , that should be apportioned to the Rydberg constant in the Balmer series if the opposite hypothesis is taken. One finds: $\delta R/R=10^{-5}$. This correction would be smaller than the difference between Rydberg constants for hydrogen and helium (1/2000), a difference which M. BOHR remarkably managed to estimate on the basis of nuclear capture. Nevertheless, given the extreme precision of spectrographic measurements, one might expect that a perturbation of electron mass due to alterations in potential energy are observable, if they exist.

4.2. Nuclear motion in atomic hydrogen

A question removed from the preceeding considerations, is that concerning the method of application of the quantum conditions to a system of charged particles in relative motion. The simplest case is that of an electron in atomic hydrogen when one takes into account simultaneous displacement of the nucleus. M. BOHR managed to treat this problem with support of the following theorem from rational mechanics: If one relates electron movement to axes fixed in direction at the centre of the nucleus, its motion is the same as for Galilean axis and as if the electron's mass equalled: $\mu_0 = m_0 M_0/(m_0 + M_0)$.

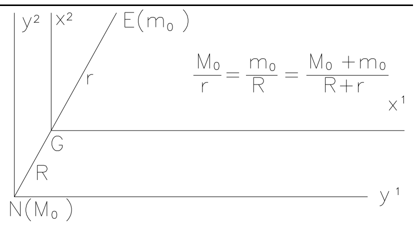

FIGURE 4.2.1. Axis system for hydrogen; y -system fixed to nucleus; x-system fixed to centre of gravity.

In a system of axis fixed in a nucleus, the electrostatic field acting on an electron can be considered as constant at all points of space, and reduced to the problem without motion of the nucleus by virtue of the substitution of the fictive mass μ_0 for the real mass m_0. In CHAPTER 2 we established a general parallelism between fundamental quantities of dynamics and wave optics; the theorem mentioned above determines, therefore, those values to be attributed to the frequency and velocity of the electronic phase wave in a system fixed to the nucleus which is not Galilean. Thanks to this artifice, quantisation conditions of stability can be considered also in this case as phase wave resonance conditions. We shall now focus on the case in which an electron and nucleus execute circular motion about their centre of gravity. The plane of these orbits shall be taken as the plane of the same two coordinates in both systems. Let space coordinates in a Galilean system attached to the centre of gravity be x_i and those attached to the nucleus be y_i; so that $x^4 = y^4 = ct$.

Let ω be the angular frequency of the line of separation of nucleus and electron about the centre of gravity G .

Further, let:

$$\eta = \frac{M_0}{M_0 + m_0}. \tag{4.2.1}$$

The transformation formulas between these two systems are then:

$$\begin{aligned} y^1 &= x^1 + R\cos(\omega t), \quad y^3 = x^3, \\ y^2 &= x^2 + R\sin(\omega t), \quad y^4 = x^4. \end{aligned} \tag{4.2.2}$$

From these equations one deduces:

$$\begin{aligned}(ds)^2 &= (dx^4)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2\\ &= \left(1-\frac{\omega^2R^2}{c^2}\right)(dy^4)^2-(dy^1)^2-(dy^2)^2-(dx^3)^2\\ &\quad -2\frac{\omega R}{c}\sin(\omega t)\,dy^1dy^4+2\frac{\omega R}{c}\cos(\omega t)\,dy^2dy^4.\end{aligned} \tag{4.2.3}$$

Components of a world momentum vector are defined by:

$$u^i=\frac{dy^i}{ds},\quad p_i=m_0cu_i+e\varphi_i=m_0cg_{ij}u^i+e\varphi_i. \tag{4.2.4}$$

One easily finds:

$$\begin{aligned}p_1 &= \frac{m_0}{\sqrt{1-\eta^2\beta^2}}\left[\frac{dy^1}{dt}+\omega R\sin(\omega t)\right],\\ p_2 &= \frac{m_0}{\sqrt{1-\eta^2\beta^2}}\left[\frac{dy^2}{dt}-\omega R\cos(\omega t)\right],\\ p_3 &= 0.\end{aligned} \tag{4.2.5}$$

Resonance of a phase wave, following ideas from CHAPTER 2, is enforced by the condition:

$$|\oint\frac{1}{h}(p_1dy^1+p_2dy^2)|=n\quad(n\text{ integer}), \tag{4.2.6}$$

where this integral is to be evaluated over the circular trajectory of the vector separation $R+r$ of the electron from the nucleus.

Since one has:

$$\begin{aligned}\frac{dy^1}{dt} &= -\omega(R+r)\sin(\omega t),\\ \frac{dy^2}{dt} &= +\omega(R+r)\cos(\omega t),\end{aligned} \tag{4.2.7}$$

if follows:

$$\frac{1}{h}\oint(p_1dy^1+P_2dy^2)=\frac{1}{h}\oint\frac{m_0}{\sqrt{1-\eta^2\beta^2}}(vdl-\omega Rvdt), \tag{4.2.8}$$

where v is the velocity of the electron with respect to the y axes and dl is the tangential infinitesimal element along the trajectory given by:

$$v=\omega(R+r)=\frac{dl}{dt}. \tag{4.2.9}$$

Finally, the resonance condition gives:

$$\frac{m_0}{\sqrt{1-\eta^2\beta^2}}\omega(R+r)\left(1-\frac{\omega R}{v}\right)\cdot 2\pi(R+r)=nh, \tag{4.2.10}$$

where, when β^2 deviates but little from 1, one gets:

$$2\pi m_0\frac{M_0}{m_0+M_0}\omega(R+r)^2=nh. \tag{4.2.11}$$

This is exactly BOHR's formula that he deduced from the theorem mentioned above and which again can be regarded as a phase wave resonance condition for an electron in orbit about a proton.

4.3. The two phase waves of electron and nucleus

In the preceeding, introduction of axes fixed on a nucleus permitted elimination of its motion, reducing the problem to an electron in an electrostatic field, thereby bringing us to the problem as treated in CHAPTER 2.

But, if we consider axes fixed with respect to the centre of gravity, both the electron and nucleus are seen to execute circular trajectories, and therefore we must consider two phase waves, one for each, and we must examine the consistency of the resulting resonance conditions.

To start, consider the phase wave of an atomic electron. In a system fixed on the nucleus, the resonance condition is:

$$\oint p_1 dy^1 + p_2 dy^2 = 2\pi\frac{m_0 M_0}{m_0+M_0}\omega(R+r)^2=nh, \tag{4.3.1}$$

where the integral is to be evaluated at a *constant time* along the circle centred at N with radius $R+r$, which is the trajectory of the relative motion and the ray of its phase wave. If now we consider the axis fixed to the centre of gravity G, the relative trajectory makes a circle centred on G of radius r; the ray of the phase wave passing through E is at *each instant* a circle centred at N and of length $R+r$, but this circle is moving because its centre is rotating about the centre of the coordinates. The resonance condition of the electron's phase wave at any given instant is not modified; it is always:

$$2\pi\frac{m_0 M_0}{m_0+M_0}\omega(R+r)^2=nh. \tag{4.3.2}$$

Consider now a phase wave of the nucleus. In all the preceeding, nucleus and electron play a symmetric role so that one can obtain the resonance condition by exchanging M_0 for m_0, and R for r; to obtain the same formulas.

In sum one sees that BOHR's conditions may be interpreted as resonance expressions for the relevant phase waves. Stability conditions for nuclear and electron motion considered separately are compatible because they are identical.

It is instructive to trace both the instantaneous positions of the two phase waves (plane features), and the trajectories as developed in the course of time (point like features) in an axes-system fixed to the centre of gravity. In such a system it appears in fact as if each moving object describes its trajectory with a velocity which at each instant is tangent to the ray of its phase wave.

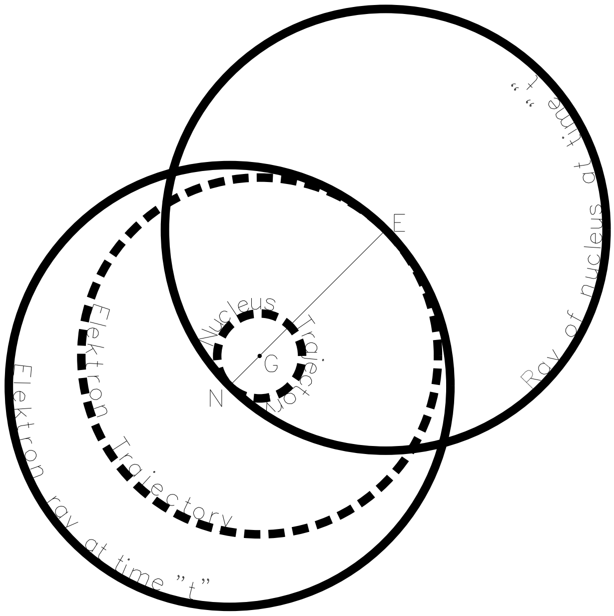

FIGURE 4.3.1. Phase rays, nucleus and electron orbits of hydrogen.

To emphasise one last point: the rays of the wave at the instant *t* are the envelopes of the velocity of propagation, but these rays are not the trajectories of energy, rather, they are their tangents at each point. This fact reminds us of certain conclusions from hydrodynamics where flow lines, envelops of velocity, are not particle trajectories when their form is invariant, in other words, if movement is constant.

CHAPTER 5

Light quanta

5.1. The atom of light[1]

As we saw in the introduction, the theory of radiation in recent times has returned to the notion of 'light particles.' A hypothetical input enabling us to develop a theory of black body radiation (as published in: "Quanta and Black Body radiation", *Journal de Physique*, Nov. 1922 — the principle results of which will be covered in CHAPTER 7) has been confirmed by the idea of real existence of "atoms of light". The concepts delineated in CHAPTER 1, and therefore the deductions made in CHAPTER 3 regarding the stability of a Bohr atom appear to be interesting confirmation of those facts leading us to form a synthesis of NEWTON's and FRESNEL's conceptions.

Without obscuring the above mentioned difficulties, we shall try to specify more exactly just how one is to imagine an "atom of light". We conceive of it in the following manner: for an observer who is fixed, it appears as a little region of space within which energy is highly concentrated and forms an indivisible unit. This agglomeration of energy has a total value ε_0 (measured by a fixed observer), from which, by the principle of inertia of energy, we may attribute to it a proper mass:

$$m_0 = \frac{\varepsilon_0}{c^2}. \tag{5.1.1}$$

This definition is entirely analogous to that used for electrons. There is, however, an essential difference between it and an electron. While an electron must be considered as a fully spherically symmetric object, an atom of light posses additional symmetry corresponding to its polarisation. We shall, therefore, represent an atom, or quantum, of light as having the same symmetry as an electrodynamic doublet. This paradigm is provisional; if accepted, still the constitution of the unit of light might be made precise only after serious modifications to electrodynamics, a task we shall not attempt here.

[1]See: EINSTEIN A., *Ann. d. Phys.,* **17**, 132 (1906); *Phys. Zeitsch.* **10**, 185 (1909).

In accord with our general notions, we suppose that there exists in the constitution of a light quantum a periodic phenomenon for which ν_0 is given by:

$$\nu_0 = \frac{1}{h} m_0 c^2. \tag{5.1.2}$$

The phase wave corresponds to the motion of this quantum with the velocity βc and with frequency:

$$\nu = \frac{1}{h} \frac{m_0 c^2}{\sqrt{1-\beta^2}}, \tag{5.1.3}$$

and it is appropriate to suppose that this wave is identical to that wave of the theory of undulation or, more exactly put, that the classical wave is a sort of a time average of a real distribution of phase waves accompanying the light atom.

It is an experimental fact that light energy moves with a velocity indistinguishable from that of the limit c. The velocity c represents a velocity that energy never obtains by reason of variation of mass with velocity, so we may assume that light atoms also move with a velocity very close to but still slightly less than c.

If a particle with an extraordinarily small proper mass is to transport a significant amount of energy, it must have a velocity very close to c; which results in the following expression for kinetic energy:

$$E = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right). \tag{5.1.4}$$

Moreover, in a very small velocity interval $(c-\varepsilon, c)$, there corresponds energies having values $(0, +\infty)$. We suppose that even with extremely small m_0 (this shall be elaborated below) light atoms still have appreciable energy and velocity very close to c; and, in spite of the virtual identity of velocities, have great variability of energy.

Since we are trying to establish a correspondence between phase waves and light waves, the frequency ν of radiation is defined by:

$$\nu = \frac{1}{h} \frac{m_0 c^2}{\sqrt{1-\beta^2}}. \tag{5.1.5}$$

We note, that we must remind ourselves that atoms of light are under consideration, the extreme smallness of $m_0 c^2$ becomes $m_0 c^2 / \sqrt{1-\beta^2}$; kinetic energy can be expressed simply as:

$$\frac{m_0 c^2}{\sqrt{1-\beta^2}}. \tag{5.1.6}$$

A light wave of frequency ν corresponds, therefore, to motion of an atom of light with velocity: $v = \beta c$ related to ν by:

$$v = \beta c = c\sqrt{1 - \frac{m_0^2 c^4}{h^2 \nu^2}}. \tag{5.1.7}$$

Except for extremely slow oscillations, $m_0 c^2/h\nu$, and *a fortiori* its square, are very small and one may pose:

$$v = c\left(1 - \frac{m_0^2 c^4}{2h^2 \nu^2}\right). \tag{5.1.8}$$

Let us try to determine the upper limit of m_0 for light. Effectively, the experiments of T. S. F.[2] have shown that even light waves with wave length of several kilometres have a velocity essentially equal to c. Let us take it that waves for which $1/\nu = 10^{-1}$ seconds have a velocity differing from c by less than 1%. This implies that the upper limit of m_0 is:

$$(m_0)_{max.} = \frac{\sqrt{2}}{10}\frac{h\nu}{c^2}, \tag{5.1.9}$$

which is approximately 10^{-24} grammes. It is possible that m_0 is still smaller; yet one might hope that some day experiments on very long wave length light will reveal evidence of a velocity discernibly below c .

One should not overlook that it is not a question regarding velocity of a phase wave, which is always *above* c , but of energy transport detectable experimentally.[3]

5.2. The motion of an atom of light

Atoms of light for which $\beta \cong 1$ are accompanied by phase waves for which $c/\beta \cong c$; that is, we think, this coincidence between light wave and phase wave is what evokes the double aspect of particle and wave. Association of FERMAT's Principle together with mechanical "least action" explains how the propagation of light is compatible with these two points of view.

Light atom trajectories are rays of their phase wave. There are reasons to believe, which we shall see below, that many light corpuscles can have the same phase wave; so that their trajectories would be various rays of the same phase wave. Thus, the old idea that a ray is the trajectory of energy is confirmed.

Nevertheless, rectilinear propagation is not a universal fact; a wave passing an edge of a screen will diffract and penetrate the shadow region; rays that pass an edge close with respect to the wave length deviate so as not to satisfy FERMAT's Principle. From a wave

[2] Changed to: 'experiments on Hertzian waves'..., in the German edition. -A.F.K.

[3] Regarding objections to these notions, see the appendix to CHAPTER 5, page 69.

point of view, this deviation results from disequilibrium introduced by a screen on various near zones of a wave. In contrast, NEWTON considered that a screen itself exercised force on light corpuscles. It seems that we have arrived at a synthesised viewpoint: wave rays curve as forseen by wave theory, but as light atoms move as if the principle of inertia of light is no longer valid, i.e., they are subject to the same motion as the phase ray to which they are unified, maybe we can say that screens exercises force on them to the extent that a curve is evidence of existence of such a force.

In the preceeding we were guided by the idea, that a corpuscle and its phase wave are not separate physical realities. Upon reflection, this seems to lead to the following conclusion: Our dynamics (in EINSTEIN's format) is based on Optics; it is a form of Geometric Optics. If it seems to us nowadays probable, that all waves transmit energy, so on the other hand, dynamics of point materiel particles doubtlessly hide wave propagation in the real sense that the principle of least action is expressible in terms of phase coherence.

It would be very interesting to study the interpretation of diffraction in space-time, but here we would encounter the problems brought up in CHAPTER 2 regarding variable motion and we do not yet have a satisfactory resolution.

5.3. Some concordances between adverse theories of radiation

Here we wish to show with some examples how the corpuscular theory of light can be reconciled with certain wave phenomena.

a.) Dopper Effect due to moving source:

Consider a source of light moving with velocity $v = \beta c$ in the direction of an observer considered to be at rest. This source emits atoms of light with frequency ν and velocity $c(1-\varepsilon)$, where $\varepsilon = m_0^2 c^4/(2h^2\nu^2)$. For a fixed observer, these quantities have magnitudes ν' and $c(1-\varepsilon')$. The theorem of addition of velocities gives:

$$c(1-\varepsilon') = \frac{c(1-\varepsilon)+v}{1+\frac{c(1-\varepsilon)v}{c^2}}, \tag{5.3.1}$$

or

$$1-\varepsilon' = \frac{1-\varepsilon+\beta}{1+(1-\varepsilon)\beta}, \tag{5.3.2}$$

where, neglecting $\varepsilon\varepsilon'$:

$$\frac{\varepsilon}{\varepsilon'} = \frac{\nu'^2}{\nu^2} = \frac{1+\beta}{1-\beta}, \quad \frac{\nu'}{\nu} = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{5.3.3}$$

if β is small, one gets the usual optics formulas:

$$\frac{\nu}{\nu} = 1+\beta, \quad \frac{T'}{T} = 1-\beta = 1-\frac{v}{c}. \tag{5.3.4}$$

It is just as easy to get the relationship between intensities measured by two observers. During a unit of time, a moving observer sees that the source emitted n photons[4] per unit of surface. The energy density of a bundle evaluated by this observer is, therfore, $nh\nu$ and the intensity is $I = nh\nu$. For a fixed observer , n photons are emitted in a a time $1/\sqrt{1-\beta^2}$ and fill a volume $c(1-\beta)/\sqrt{1-\beta^2} = c\sqrt{(1-\beta)/(1+\beta)}$. Thus, the energy density of a bundle appears to be:

$$\frac{nh\nu}{c}\sqrt{\frac{1+\beta}{1-\beta}}, \tag{5.3.5}$$

and the intensity:

$$I' = nh\nu'\sqrt{\frac{1+\beta}{1-\beta}} = nh\nu'\frac{\nu'}{\nu}. \tag{5.3.6}$$

From which we get:

$$\frac{I'}{I} = \left(\frac{\nu'}{\nu}\right)^2. \tag{5.3.7}$$

All these formulas from a wave point of view can be found in[5].

b.) Reflection from a moving mirror.

Consider reflection of a photon impinging perpendicularly on a mirror moving with velocity βc in a direction perpendicular to its surface.

For an observer at rest, ν' is the frequency of phase waves accompanying photons with velocity $c(1-\varepsilon_1')$. For a stationary observer, this frequency and velocity are: ν_1 and $c(1-\varepsilon_1)$.

If we now consider reflected photons, their corresponding values are: ν_2, $c(1-\varepsilon_2)$, ν_2' and $c(1-\varepsilon_2')$.

The addition law for velocities gives:

$$c(1-\varepsilon_1) = \frac{c(1-\varepsilon_1') + \beta c}{1+\beta(1-\varepsilon_1')},$$

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[4]Note that DE BROGLIE's term was "light atom" or "quantum", not "photon", a term coined by G. N. LEWIS first a year later in 1926. For the sake of contemporary readability, however, hereafter in this translation the latter term is used. -A.F.K.

[5]VON LAUE, M., *Die Relativitätstheorie,* Vol. I, 3 ed. p. 119.

(5.3.8) $$c(1-\varepsilon_2) = \frac{c(1-\varepsilon_2') - \beta c}{1-\beta(1-\varepsilon_2')}.$$

For a stationary observer, reflection occurs without change of frequency because of conservation of energy. That is:

(5.3.9) $$\nu_1 = \nu_2, \quad \varepsilon_1 = \varepsilon_2, \quad \frac{1-\varepsilon_1'+\beta}{1+\beta(1-\varepsilon_1')} = \frac{1-\varepsilon_2'-\beta}{1-\beta(1-\varepsilon_2')}.$$

Neglecting $\varepsilon_1'\varepsilon_2'$, gives:

(5.3.10) $$\frac{\varepsilon_1'}{\varepsilon_2'} = \left(\frac{\nu_2'}{\nu_1'}\right)^2 = \left(\frac{1+\beta}{1-\beta}\right)^2.$$

If β is small, one recovers the classical formula:

(5.3.11) $$\frac{T_2}{T_1} = 1 - 2\frac{v}{c}.$$

Oblique reflection is easily included.

Let n be the number of photons reflected during a given time interval. Total energy of these n photons after reflection, E_2', is in proportion to their energy before reflexion, E_1', given by:

(5.3.12) $$\frac{nh\nu_2'}{nh\nu_1'} = \frac{\nu_2'}{\nu_1'}.$$

Although Electrodynamics also yields this relation, here its derivation is absolutely transparent.

If n photons occupy a volume V_1 before reflexion, the volume after equals: $V_2 = V_1(1-\beta)/(1+\beta)$, which elementary geometric reasoning shows easily. The ratio of intensities before and after reflexion is given by:

(5.3.13) $$\frac{I_2'}{I_1'} = \frac{nh\nu_2'}{nh\nu_1'}\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{\nu_2'}{\nu_1'}\right)^2.$$

All these results are also given in[6].

c.) Black body radiation pressure:

Consider a cavity filled with black body radiation at temperature T. What is the pressure on the cavity walls? In our view, black body radiation is a photon gas with, we presume, an isotropic distribution of velocities. Let u be the total energy (or, what is here the same, total kinetic energy) of the photons in a unit volume. Let ds be an infinitesimal wall element, dv , a volume element, r its distance from the coordinate origin, and θ the

[6] VON LAUE, M. *Electrodynamik*, p.124.

angle to the normal of the wall. The solid angle under which the element ds is seen from the centre O of dv is:

$$d\Omega = \frac{ds\cos\theta}{r^2}. \tag{5.3.14}$$

Consider now only those photons in a volume dv whose energy is between w and $w+dw$, in quantity: $n_w dwdv$; the number among them which are directed toward ds is, by virtue of isotropy, equal to:

$$\frac{d\Omega}{4\pi} \times n_w dwdv = n_w dw \frac{ds\cos\theta}{4\pi r^2} dv. \tag{5.3.15}$$

Changing to polar coordinates with the normal to ds as polar axis, one finds:

$$dv = r^2 \sin\theta\, d\theta d\psi dr. \tag{5.3.16}$$

Moreover, kinetic energy of a photon would be $m_0c^2/\sqrt{1-\beta^2}$ and its momentum $G = m_0 v/\sqrt{1-\beta^2}$, so that when $v \to c$ one gets:

$$\frac{W}{c} = G. \tag{5.3.17}$$

Thus, by reflection at angle θ of a photon of energy w, an impulse $2G = 2W\cos\theta/c$ is imparted to ds; i.e., photons in dv impart an impulse to ds through reflection of :

$$2\frac{W}{c}\cos\theta\, n_w dwr^2 \sin\theta\, d\psi dr \frac{ds\cos\theta}{4\pi r^2}. \tag{5.3.18}$$

Integrating now first with respect to w from 0 to ∞ and noting that $\int_0^\infty wn_w dw = u$, then with respect to ψ and θ from 0 to $2\pi$and 0 to $\pi/2$ respectively, and finally r from 0 to c, we obtain the total momentum deposited in one second on ds and, by dividing by ds, we obtain an expression for radiation pressure:

$$p = u\int_0^{\pi/2} \cos^2\theta \sin\theta d\theta = \frac{u}{3}. \tag{5.3.19}$$

Radiation pressure, we find, equals one third of the energy contained in a unit volume, a result which is the same as that obtained from classical theory.

The ease with which we recovered certain results known from wave theory reveals the existence between two apparently opposite points of view of a concealed harmony that nature presents via phase waves.

5.4. Photons and wave optics[7]

The keystone of the theory of photons is its explanation of wave optics. The essential point is that this explanation necessitates introduction of a phase wave for periodic phenomena; it seems we have managed to establish a close association between the motion of photons and wave propagation of a particular mode. It is very likely, in effect, that if the theory of photons shall explain optical wave phenomena, most likely it will be done with notions of this type. Unfortunately, it is still not possible to claim satisfactory results for this task; the most we can say is that EINSTEIN's audacious discrete photon conception was judiciously applied so as to encompass a number of phenomena which in the XIX century were considered to so convincingly have verified the continuous wave theory.

Let us turn now to attacking this difficult problem on the flanks. To proceed at this task, it is necessary, as we said, to establish a certain natural liason, no doubt of a statistical character, between classical waves and the superposition of phase waves; which should lead inexorably to attribute an electromagnetic character to phase waves so as to account for periodic phenomena, as delineated in CHAPTER 1.

On can consider it proven with near certitude, that emission and adsorbtion of radiation occurs in a discontinuous fashion. Electromagnetism, or more precisely the theory of electrons, gives a rather inexact explication of these processes. However, M. BOHR, with his correspondence principle, has shown us that if one attributes the assumptions of this theory to an ensemble of electrons, then it has a certain global exactitude. Perhaps all of electrodynamics has only a statistical validity; MAXWELL's equations then are a continuous approximation of discrete processes, just as the laws of hydrodynamics are a continuous approximation to the complex detailed motion of molecules of a fluid. This correspondence being sufficiently imprecise and elastic, can serve as guidance for intrepid researchers who wish to find a theory of electromagnetism in better accord with the concept of photons.

In the next section we shall develop our ideas on interference; in all candour, however, they should be taken as speculations more than explanations.

5.5. Interference and coherence[8]

Let us consider how to establish the presence of light at a point in space. To start, one places a material object with which light reacts either chemically, thermally, etc., at

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[7]See: BATEMAN, H., "On the theory of light Quanta," *Phil. Mag.,* **46** (1923), 977 for historical background and bibliography.

[8]Footnote in the German translation: In more recent work, the author proposed a different theory of interference. (See: *Comptes Rendus,* **183**, 447 (1926).)

this point; it is possible that in the last analysis all of these effects are just the photoelectric effect. One can also consider the diffusion of waves at this point in space. Thus, one can say that where there is no such reaction with material, light would be undetectable experimentally. Electromagnetic theory holds that photographic effects (e.g., WIENER's experiments), and diffusion, occur in proportion to the electric field intensity; wherever the electric field intensity is null, even if there is magnetic energy, these effects are indiscernible.

The ideas developed herein lead to associating phase waves with electromagnetic waves; at least regarding phase wave distribution in space; however, questions regarding intensities must be set aside. This notion together with that of the correspondence leads us to consider that the probability of an interaction between material particles and photons at each point in space depends on the intensity (more accurately on its average) of a vector characteristic of the phase wave, and where this is null, there is no detection—there is negative interference. One imagines, therefore, that where photons traverse an interference region, they can be absorbed in some places but not in others. This is in principle a very qualitative explanation of interference, while taking the discontinuous feature of light energy into account. M. NORMAN CAMPBELL in his book "Modern Electrical Theory" (1913) appears to have envisioned a solution of the same ilk when he wrote: "Only the corpuscular theory of light can explain how energy is transfered at a spot, while only the wave theory can explain why the transfer along a trajectory depends on location. It seems that energy itself is transported by particles while its absorption is determined by special waves".

So that interference can produce extended spacial patterns, it seems necessary that various atoms within a source be coordinated. We propose to express this coordination by means of the following principle: *A phase wave passing through material bodies induces them to emit additional photons whose phase wave is identical to that of the stimulus.* A wave, therefore, can consist of many photons that retain the same phase. When the number of photons is very large, this wave very closely resembles the classical conception of a wave.

5.6. BOHR's frequency law and Conclusions

Whatever point of view one adapts, details of the internal transformations that a material atom undergoes by emission and absorbtions can not be imagined at all. Although we steadfastly retain the granular hypothesis, we do not know in the least if a photon adsorbed by an atom is stored within it, or if the two meld into a unified entity. Likewise, we do not know if emission is ejection of a preexisting photon or the creation of a new one from internal energy. Whatever the case, it is certain that emission never results in less than a single quantum; for which the total energy equals h times the frequency of the photon's accompanying phase wave; to salvage the conservation of energy principle, it must be taken that emission results in the diminution of the source atom's internal energy in accord with BOHR's Law of frequencies*:*

$$h\nu = W_1 - W_2. \tag{5.6.1}$$

One sees that our conceptions, after having leads us to a simple explanation of stability conditions, leads also to the Law of Frequencies, *if we impose the condition that an emission always comprises just one photon.*

We note that the image of emission from the quantum theory seems to be confirmed by the conclusions of MM. EINSTEIN and LEON BRILLOUIN,[9] which showed the necessity to introduce into the analysis of the interaction of black body radiation and a free body the idea that emission is precisely directed.

How might we conclude this chapter? Surely, although those phenomena such as dispersion appear incompatible with the notion of photons, at least in its simple form, it appears that now they are less inexplicable given ideas regarding phase waves. The recent theory of X and γ-ray diffusion by M. A.-H. COMPTON, which we shall consider below, supports the existence of photons with substantial empirical evidence in a domain in which formerly the wave notion reigned supreme. It is, nonetheless, incontestable that concepts of parcelled light energy do not provide any resolution in the context of wave optics, and that serious difficulties remain; it is, it seems to us, therefore, premature to judge its final fate.

[9]**18**, 121 (1917); BRILLOIN, L., *Journ. d. Phys., série* **VI**, 2, 142 (1921).

CHAPTER 6

X and γ-ray diffusion

6.1. M. J. J. THOMSON's theory

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In this chapter we shall study X- and γ-ray diffusion and show by suggestive examples the respective views supported by electromagnetic and photon theory.

Let us begin by defining the phenomenon of diffusion, according to which one envisions a bundle of rays, some of which are scattered in various directions. On says that there is diffusion if the bundle is weakened by redirecting some rays while traversing material.

Electron theory explains this quite simply. It supposes (in direct opposition to BOHR's atomic model) that electrons in atoms are subject to quasi-elastic forces and have determined frequencies, so that passage of an electromagnetic wave affects the amplitude of the oscillation of the electrons depending on the frequencies of both electrons and wave. In conformity with the theory of wave generation, electron motion is ceaselessly diminished by emission of a cylindrical waves. This eventually establishes equilibrium between the incident and redirected radiation. The final result is that there is a scattering of a fraction of the incident waves into all directions in space.

In order to calculate the extent of diffusion, the motion of such oscillating electrons must be determined. To do so one may express equilibrium between the resulting inertial force and the quasi-elastic force on the one hand and the electric force from the impinging radiation on the other. In the visible range, numerical results show that the inertial term can be neglected in comparison to the quasi-elastic term, so an amplitude proportional to that of the stimulus wave, but independent of its frequency, can be attributed to the electronic oscillation. The theory of dipole radiation shows that the intensity of secondary radiation falls off as the fourth power of wave length, so that waves are diffused more strongly as their frequency increases. This is the theory with which Lord RAYLEIGH explained the blue colour of the sky.[2]

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1 THOMSON, J. J., *Passage de l'électricité à travers les gaz*, (Gauthier-Villars, Paris, 1912) p. 321.

2 the basis of the elastic theory of light, but this explanation accords well with electromagnetic theory also.

In the high frequency X and γ-ray region, it is, contrariwise, the quasi-elastic term, in comparison to the inertial term, that is negligible. All transpires as if electrons were free and vibrational motion simply proportional not only to the incident amplitude, but also now to wave length squared. This was the empirical base that led J. J. THOMSON to formulate the original theory of X-ray diffusion. These two principles can be stated as follows:

1° If one designates by θ the diffusion angle relative to the incidence direction, energy as a function of θ is given then by $(1+\cos\theta)/2$.

2° The ratio of diffused to incident energy per second is given by:

$$\frac{I_\alpha}{I} = \frac{8\pi}{3}\frac{e^4}{m_0^2 c^4}, \tag{6.1.1}$$

where e and m_0 pertain to the electron and c is the speed of light.

An atom certainly contains more that one electron; nowadays there is good reason to suppose that the number of electrons in an element equals its atomic number. M. THOMSON assumed *incoherent* emission from the p electrons in an atom; and, therefore, considered that the diffused energy should be p times that of a single electron. According to empirical evidence, diffusion suffers a gradual diminution given by an exponential law:

$$I_x = I_0 e^{-sx}, \tag{6.1.2}$$

where s is the decay or 'diffusion' constant. This constant, normalised by material density, s/ρ, is the bulk diffusion constant. If one denotes the 'atomic' diffusion constant σ as that relative to a single atom, then in terms of bulk diffusion constant it would be:

$$\sigma = \frac{s}{\rho} A m_H, \tag{6.1.3}$$

where A is the atomic number of the scatter and m_H is the mass of hydrogen. Substituting the numerical factor from Eq. (6.1.1), one gets:

$$\sigma = 0.54 \times 10^{-24} p. \tag{6.1.4}$$

But, experience shows that s/ρ is very nearly 0.2, so that one has:

$$\frac{A}{p} = \frac{0.54 \times 10^{-24}}{0.2 \times 1.46 \times 10^{-24}} = \frac{0.54}{0.29}. \tag{6.1.5}$$

This quantity is nearly 2, which accords well with our notion of the ratio of the number of electrons to atomic weight. Thus, M. THOMSON's theory leads to interesting coincidences with various experiments, notably M. BARKLA's, which, largely, have been verified already long ago[3]

6.2. DEBYE's theory[4]

There remains difficulties; in particular, M. W. H. BRAGG has found a stronger diffusion than calculated above for which he concludes that the dispersed energy is proportional not to the number of scattering centres, but to its square. M. DEBYE has proposed a theory completely compatible with both MM. BRAGG and BARKLA.

M. DEBYE considers that the atomic electrons are distributed regularly in a volume with dimensions of the order of 10^{-8} cm.; for the sake of calculations he supposes they are distributed on a circle. If the wave length is long with respect to the average distance between electrons, the motion of the electrons will be essentially in phase and, for the whole wave the amplitudes of each ray add. The diffused energy then is proportional to p^2, and not p , so that σ becomes:

$$\sigma = \frac{8\pi}{3}\frac{e^4}{m_0^2c^4}p^2. \tag{6.2.1}$$

So, with respect to spacial distribution, it is identical to M. THOMSON's result.

For waves with progressively shorter wave lengths, the spacial distribution is asymmetric, energy in the direction from which it came is less than in the opposite direction. The reason for this is: one may no longer regard the vibrations of the various electrons as being in phase when the wave length is comparable to interatomic distances. The amplitudes of rays in various directions do not add because they are out of phase and therefore diffused energy is reduced. However, in a sharp cone in the direction of propagation, they are in phase so that amplitudes add and diffusion within the cone is much stronger than elsewhere. M. DEBYE was first to observe a curious phenomenon, when diffused energy is charted along the axis of the cone defined above, intensity is not regular but, shows certain periodical variations; on a screen placed perpendicular to the propagation direction one sees concentric bright rings cantered on the axis. Even though M. DEBYE believes he has seen this phenomenon in certain experiments done by M. FRIEDRICH, it seems that so far there is no real confirmation.

For short wave lengths, this phenomenon can be simplified. The strong diffusion cone recedes progressively, the distribution reverts to being symmetric and begins to satisfy THOMSON's formulas because the waves from various electrons are no longer in phase, so it becomes energies that add, not amplitudes.

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[3]by MM. R. LEDOUX-LEBARD and A. DAUVILLIER, *La physique des rayons X* (Gauthier-Villars, Paris, 1921) p. 137.

[4]DEBYE, P., *Ann. d. Phys.*, **46**, 809 (1915).

The great advantage of M. DEBYE's theory is that it explained the strong diffusion of soft X-rays and showed how it happens that when frequency increases the theory goes over to THOMSON's. But it is essential to note that following DEBYE's ideas, the higher the frequency, the more symmetric diffused radiation, so that the value 0.2 of the coefficient s/ρ can be obtained. However, we shall see in the following section, that this is not at all the case.

6.3. The recent theory of MM. DEBYE and COMPTON

Experimentation with X and γ-rays reveals facts quite distinct from those predicted by the above theory. To begin, the higher the frequency, the more pronounced the dissymmetry of diffused radiation; on the other hand, the less the total diffused energy, the more the value of the coefficient s/ρ decreases rapidly until the wave length goes under 0.3 or 0.2$\AA$ and becomes very weak for γ-rays. So, there where THOMSON's theory should apply better, in fact it applies less.

Two additional light phenomena have been discover recently by clever experimentation, including that by M. A. H. COMPTON. One is that it appears that diffusion in the direction of the stimulus-radiation is accompanied by a reduction of frequency; and, the other is ejection of the scattering electron. Practically simultaneously both MM. P. DEBYE and A. H. COMPTON, each in his own way, have found an explanation for these phenomena based on classical physics principles and the existence of photons.

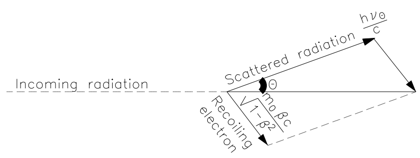

FIGURE 6.3.1. Compton scattering

Their idea is: if a photon passes close enough to an electron, it can be taken that they interact. Before completion of an interaction an electron absorbs a certain amount of energy from a photon so that after interaction the frequency of a photon is reduced such that the outcome is governed by conservation of momentum. Suppose that a scattered photon goes in a direction at angle θ to incoming radiation. Frequencies before and after interaction are ν_0 and ν_θ and proper mass of an electron is m_0, so that one has:

$$h\nu_\theta = h\nu_0 - m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right), \tag{6.3.1}$$

$$\left(\frac{m_0\beta c}{\sqrt{1-\beta^2}}\right)^2 = \left(\frac{h\nu_0}{c}\right)^2 + \left(\frac{h\nu_\theta}{c}\right)^2 - 2\frac{h\nu_0}{c}\frac{h\nu_\theta}{c}\cos\theta. \tag{6.3.2}$$

Eq. (6.3.2) was derived with aid of Fig. (6.3.1).

The velocity $v = \beta c$ is the velocity an electron acquires during the interaction.

Let α be the ratio $h\nu_0/m_0c^2$, which is equal to the quotient of ν_0 and an electron's proper frequency, so that it follows:

$$\nu_\theta = \frac{\nu_0}{1+2\alpha\sin^2(\theta/2)}, \tag{6.3.3}$$

or

$$\lambda_\theta = \lambda_0\left(1+2\alpha\sin^2(\theta/2)\right). \tag{6.3.4}$$

With aid of these formulas one can study speed and direction of photon scatter as well as electron 'kick back' or recoil. One finds that photon scattering direction varies from 0 to π and that electron recoil from $\pi/2$ to 0, while its velocity will be between 0 and a certain maximum.

M. COMPTON, appealing to an hypothesis inspired by the correspondence principle, seems to have calculated scattered energy and thereby explained the rapid diminution of the coefficient s/ρ. M. DEBYE applied the correspondence principle somewhat differently but obtained an equivalent interpretation of the same phenomenon.

In an article in *The Physical Review* (May, 1923) and in another more recent article in the *Philosophical Magazine* (Nov. 1923), M. A. H. COMPTON shows how these ideas enable computation of many experimental facts, in particular for hard rays in soft materials; the variation of wave length has been quantitatively verified. For solid bodies and soft radiation, it seems that there coexists a diffused line with no change of frequency and another diffused line which follows the Compton-Debye law. For low frequencies the first appears to predominate to the extent that, that is all there is. Experiments by M. ROSS on scattering of MoK_α and green light in paraffin confirms this point of view. K_α lines exhibit a strong line of scattered radiation following COMPTON's Law and a weak line of unaltered frequency, which appears to be true only for green light.

The existence of a non displaced line appears to explain why scattering in a crystal (von Laue effect) is not accompanied by a variation of wave length. MM. JAUNCEY and WOLFERS have shown recently that, in effect, the lines of scattered radiation from the crystals usually used as reflectors, would exhibit to an appreciable extent the Compton-Debye effect; in fact, measurements of the wave length of Röntgen waves have shown this effect. It must be taken that in this case scattering occurs without photon degradation.

To begin, let us try to explain these two types of scattering in the following manner: the Compton effect occurs whenever an electron is relatively weakly bound in a scattering material, the other case, on the other hand, occurs when incident photons suffer little change in wave length because the scattering centre can not respond and compensate by virtue of its high inertia. M. COMPTON had difficulties accepting this explanation, preferring to consider that multiple scatterings of the outgoing photon were involved, thereby making evident the sum of masses of all scattering centres. Whichever way it is, one must admit that hard photons and heavy scatters behave differently that soft photons and light scatters.

As a means to render the conception of scattering as being the deviation of photons compatible with conservation of phase—as found necessary to explain VON LAUE's interference patterns, this explanation is subject to the difficulties considered above, and not at all further clarified from the point of view of wave optics that was considered in the preceeding section.

Up to the matter of hard X-rays and light materials, as they are in practice for radiotherapy, these phenomena must be completely modified by COMPTON's effect, and it appears that, that is indeed what happens. We shall now give an example. One knows that there is a greater diminution by scattering suffered by a sheaf of X-rays traversing material than by absorption, a phenomenon that is accompanied by emission of photoelectrons. An empirical law by MM. BRAGG and PIERCE shows that this absorption varies as the cube of the wave length and undergoes distinct discontinuities for each characteristic frequency of the interatomic levels of the considered substance; moreover, for the same wave length and diverse elements, the coefficient of atomic absorption varies as the fourth power of the atomic number.

This law is well verified in the middle range of Röntgen frequencies and it seems highly probable that it will apply as well to hard X-rays. In so far as, following the conceptions from the Compton-Debye theory, scattering is exclusively wave scattering, only the absorbed energy following the Bragg Law can produce ionisation of the gas by high velocity photoelectrons shocking atoms. The Bragg-Pierce Law then permits calculating the ionisation produced by the same hard radiation in two separate ampoules, one with a heavy gas, for example CH^3I, and the other with a light gas such as air. Even if various ancillary corrections are taken into account, this result is seen experimentally to be much smaller than calculations predict. M. DAUVILLIER has observed this phenomenon in X-rays for which an explanation is for me an old intriguing question.

The new scattering theory appears to be able to explain these anomalies quite well. When, in effect, at least in the case of hard radiation, a portion of the energy is transfered to scattered electrons, there is not only scattering of radiation but also “absorption by scattering”. Ionisation in the gas is due to both electrons being stripped from atoms as well as by recoil of electrons. In a heavy gas (CH^3I), Bragg absorption in comparison to Compton absorption is strong. For a light gas (air), it is not the same; the first cause of this due to variation by N^4 is very weak and the second dependant on N should be the more important one. Total adsorbtion, and therefore ionisation in the two gases, should therefore be much smaller than anticipated. It is possible in this way even to compute the ionisation. One sees with this example the large practical consequence of the ideas of MM. COMPTON AND DEBYE. Recoil of the scattered electrons also provides the key idea to understanding many other phenomena.

6.4. Scattering via moving electrons

One can generalise the Compton-Debye theory by considering scattering of photons off moving electrons. Let us take the x axis to be the direction of incoming photons whose frequency is ν_1, the y and z axes may be arbitrarily chosen to be orthogonal and in a plane containing the scattering centre. The direction of the velocity, βc, of the electron *before* impact of the photon is defined by the direction cosines a_1, b_1, c_1, and we let θ_1 be the angle with the x axis, i.e., $a_1 = \cos\theta_1$; after the impact, a scattered photon propagates with frequency ν_2 and with direction cosines p, q, r making an angle φ with the initial electron velocity ($\cos\varphi = a_1 p + b_1 q + c_1 r$) and the angle θ with the x axis ($p = \cos\theta$). Let the electron's have final velocity $\beta_2 c$ whose direction cosines are a_2, b_2, c_2.

Conservation of energy and momentum during the impact imply:

$$h\nu_1 + \frac{m_0c^2}{\sqrt{1-\beta_1^2}} = h\nu_2 + \frac{m_0c^2}{\sqrt{1-\beta_2^2}}, \tag{6.4.1}$$

$$\frac{h\nu_1}{c} + \frac{m_0\beta_1 c}{\sqrt{1-\beta_1^2}}a_1 = \frac{h\nu_2}{c}p + \frac{m_0\beta_2 c}{\sqrt{1-\beta_2^2}}a_2, \tag{6.4.2}$$

$$\frac{m_0\beta_1 c}{\sqrt{1-\beta_1^2}}b_1 = \frac{h\nu_2}{c}q + \frac{m_0\beta_2 c}{\sqrt{1-\beta_2^2}}b_2, \tag{6.4.3}$$

$$\frac{m_0\beta_1 c}{\sqrt{1-\beta_1^2}}c_1 = \frac{h\nu_2}{c}r + \frac{m_0\beta_2 c}{\sqrt{1-\beta_2^2}}c_2. \tag{6.4.4}$$

Eliminate a_2, b_2, c_2 using $a_2^2 + b_2^2 + c_2^2 = 1$; then, from the resulting equations and those expressing the conservation of energy, eliminate β_2. Now, with COMPTON's relationship: $\alpha = h\nu_1/m_0c^2$, it follows that:

$$\nu_2 = \nu_1 \frac{1-\beta_1\cos\theta_1}{1-\beta_1\cos\varphi + 2\alpha\sqrt{1-\beta_1^2}\sin^2(\theta/2)}. \tag{6.4.5}$$

When the initial velocity is null or negligible, we get COMPTON's formula:

$$\nu_2 = \nu_1 \frac{1}{1+2\alpha\sin^2(\theta/2)}. \tag{6.4.6}$$

In the general case, the Compton effect, represented by the term with α, is present but diminished; moreover, the Doppler Effect also arises. If the Compton Effect is negligible, one finds:

$$\nu_2 = \nu_1 \frac{1-\beta_1\cos\theta_1}{1-\beta_1\cos\varphi}. \tag{6.4.7}$$

As, in this case, photon scattering does not disturb electron motion, one might expect to get a result identical to that from electrodynamics. This is effectively what happens. Let us calculate now the frequency of the scattered radiation (including relativistic effects). The impinging radiation with respect to the electron has the frequency:

$$\nu' = \nu_1 \frac{1-\beta_1\cos\theta_1}{\sqrt{1-\beta_1^2}}.$$

If the electron maintains its translation velocity $\beta_1 c$, it will start to vibrate at frequency ν', an observer who receives radiation scattered in the direction making an angle φ with respect to $\beta_1 c$ of the source, attributes the frequency:

$$\nu_2 = \nu' \frac{\sqrt{1-\beta_1^2}}{1-\beta_1\cos\varphi}, \tag{6.4.8}$$

from which one easily gets:

$$\nu_2 = \nu_1 \frac{1-\beta_1\cos\theta_1}{1-\beta_1\cos\varphi}. \tag{6.4.9}$$

The Compton Effect remains in general quite weak, while the Doppler Effect attendant to a fall of several hundred kilovolts can reach high values (an increase of a third for 200 kilovolts).

Here we have to do with a strengthening of the photon because the scattering electron is itself moving with high velocity and gives some of its energy to the radiation. The conditions for STOKE's Law are not met. It is not impossible that some of the above conclusions could be verified experimentally, at least those concerning X-rays.

CHAPTER 7

Quantum Statistical Mechanics

7.1. Review of statistical thermodynamics

The interpretation of the laws of thermodynamics using statistical considerations is one of the most beautiful achievements of scientific thought, but it is not without its difficulties and objections. In is not intended in the context of this work to analyse critically these methods; we intend here first to recall certain fundamentals in their currently common form, and then examen how they affect our new ideas for the theory of gases and black body radiation.

BOLTZMANN has shown, to begin, that the entropy of a gas in a particular state is, up to an arbitrary additive constant, the product of the logarithm of the probability of the state times "BOLTZMANN's constant" k, which depends on the temperature scale; he arrived at this notion for the first time from analysis of the random collisions of the gas molecules. Nowadays, from the works of MM. PLANCK and EINSTEIN one prefers the relationship $S = k \log P$ as the definition of the system's entropy S. In this definition, P is not the mathematical probability equal to the number of micro-configurations giving the same macroscopic configuration over the total number of possible configurations, rather the "thermodynamic probability" equal simply to the numerator of this ratio. This choice of definition for P allows for the determination (albeit somewhat arbitrarily) of the constant of entropy. These postulates recall a well known demonstration of a certain analytic derivation of thermodynamic quantitates that has the advantage of being applicable to the case of continuously variable states, as well as discontinuously variable ones.

Consider $\aleph$ objects which may distributed among m "states" or "cells" considered *a priori* to be equally probable. A certain configuration is realized when there are n_1 objects in cell 1, n_2 in cell 2, etc. The thermodynamic probability then would be:

(7.1.1) $$p = \frac{\aleph!}{n_1!n_2!\ldots n_n!}.$$

When $\aleph$ and all the n_i are large numbers, we may use STIRLING's formula to obtain the system entropy:

$$S = k \log P = k\aleph \log \aleph - k\sum_1^m n_i \log n_i. \tag{7.1.2}$$

Suppose that for each cell there corresponds a value of a function which we shall call the "energy of an object in that cell". Now we consider the resorting of objects among cells such that the total energy is left unaltered. Entropy will then vary as:

$$\delta S = -k\delta\left[\sum_1^m n_i \log n_i\right] = -k\sum_1^m \delta n_i - k\sum_1^m \log n_i \delta n_i, \tag{7.1.3}$$

with the conditions: $\sum_1^m \delta n_i = 0$ and $\sum_1^m \varepsilon_i \delta n_i = 0$. Maximum entropy is determined by the condition: $\delta S = 0$. The method of indeterminate coefficients requires that, to satisfy the minimum condition, the following equation must be satisfied:

$$\sum_1^m [\log n_i + \eta + \beta_i \varepsilon_i]\, \delta n_i = 0, \tag{7.1.4}$$

where η and β, as well as the δn_i, are constants. Given the above, one concludes that the most probable distribution, the only one realized for all practical purposes, is:

$$n_i = \alpha e^{-\beta\varepsilon_i}, \quad (\alpha = e^{-\eta}). \tag{7.1.5}$$

This is the so-called "canonical" distribution. The thermodynamic entropy of the system corresponding to this most probable distribution, is given by:

$$S = k\aleph \log \aleph - \sum_1^m \left[k\alpha e^{-\beta\varepsilon_i}(\log\alpha - \beta\varepsilon_i)\right]; \tag{7.1.6}$$

however, while $\sum_1^m n_i = \aleph$ and $\sum_1^m \varepsilon_i n_i = E$ (total energy), Eq. (7.1.6) is alternately:

$$S = k\aleph \log \frac{\aleph}{\alpha} + k\beta E = k\alpha \log \sum_1^m e^{-\beta\varepsilon_i} + k\beta E. \tag{7.1.7}$$

To determine β we use the thermodynamic relations:

$$\frac{1}{T} = \frac{dS}{dE} = \frac{\partial S}{\partial\beta}\cdot\frac{\partial\beta}{\partial E} + \frac{\partial S}{\partial E} \tag{7.1.8}$$

$$= -k\aleph \frac{\sum_1^m \varepsilon_i e^{-\beta\varepsilon_i}}{\sum_1^m e^{-\beta\varepsilon_i}}\frac{d\beta}{dE} + kE\frac{d\beta}{dE} + k\beta, \tag{7.1.9}$$

and because

$$\aleph \frac{\sum_1^m \varepsilon_i e^{-\beta\varepsilon_i}}{\sum_1^m e^{-\beta\varepsilon_i}} = \aleph\bar{\varepsilon} = E, \tag{7.1.10}$$

it follows that

$$\frac{1}{T} = k\beta, \quad \beta = \frac{1}{kT}. \tag{7.1.11}$$

The free energy can be calculated from:

$$\begin{aligned} F = E - Ts &= E - k\aleph T \log\left[\sum_1^m e^{-\beta\varepsilon_i}\right] - \beta kTE \\ &= k\aleph T \log\left[\sum_1^m e^{-\beta\varepsilon_i}\right]. \end{aligned} \tag{7.1.12}$$

The mean value of the free energy for one of the objects is therefore:

$$\bar{F} = -kT \log\left[\sum_1^m e^{-\beta\varepsilon_i}\right]. \tag{7.1.13}$$

Let us apply these considerations to a gas comprised of identical molecules of mass m_o. From LIOUVILLE's Theorem (equally valid in relativistic dynamics) we learn that the infinitesimal element of phase space for a molecule, $dxdydzdpdqdr$ (where x, y, z are coordinates of position and p, q, r are the components of momentum), is an invariant of the equations of motion and therefore its value is independent of the choice of coordinates. From this one is led to the idea that the number of equally probable states is also proportional to this quantity. In turn, one is then led to MAXWELL's Equal Partition Law giving the number of atoms falling in the element $dxdydzdpdqdr$:

$$dn = Ce^{-\frac{w}{kT}} dxdydzdpdqdr, \tag{7.1.14}$$

where w is the kinetic energy of the atoms.

Suppose that the velocities are sufficiently weak to justify using nonrelativistic dynamics, then we have:

$$w = \frac{1}{2}m_0 v^2, \quad dpdqdr = 4\pi G^2 dG, \tag{7.1.15}$$

where $G = m_0 v = \sqrt{2m_0 w}$ is the momentum. Finally, the number of atoms contained in this volume element, whose energy is between w and $w + dw$ is given by the classical formula:

$$dn = Ce^{-\frac{w}{kT}} 4\pi m_0^{3/2} \sqrt{2w} dwdxdydz. \tag{7.1.16}$$

It remains now only to calculate the free energy and entropy. To do so we take as the object of the general theory, not an isolated molecule, but a gas comprised of N identical molecules of mass m_0 such that the state is defined by $6N$ parameters. Free energy of this gas in the thermodynamical sense is defined following GIBBS, as the average volume of the $N-$ atom gas, it would be:

$$\bar{F} = -kT\log\left[\sum_{1}^{m} e^{-\beta\varepsilon_i}\right], \quad \beta = \frac{1}{kT}. \tag{7.1.17}$$

M. PLANCK has shown how this sum is to be evaluated: it is to be expressed as an integral over the the phase space of $6N$ dimensions, which is equivalent to the product of N integrals over the phase space of a single molecule, but divided by $N!$ to take account of indistinguishability of molecules. Free energy can be calculated in a similar fashion; from this one gets the entropy and energy from the usual thermodynamic relationships:

$$S = -\frac{\partial F}{\partial T}, \quad E = F + TS. \tag{7.1.18}$$

In order to do these calculations, it is necessary to determine the constant C in Eq. (7.1.16). This factor has dimensions of inverse cube of action. M. PLANCK has determined it with the following disconcerting hypothesis: "*Phase space for a molecule is divided into cells of volume* h^3". One can say, therefore, that in each cell there is a single point whose probability is not null, or that all points of the same cell correspond to states impossible to distinguish physically from each other.

PLANCK's hypothesis leads to writing free energy as:

$$\begin{aligned} F &= -kT\log\left[\frac{1}{N!}\left(\int\int\int\int\int\int_{-\infty}^{+\infty} e^{-\frac{\varepsilon}{kT}}\frac{dxdydzdpdqdr}{h^3}\right)^N\right] \\ &= -NkT\log\left[\frac{e}{N}\left(\int\int\int\int\int\int_{-\infty}^{+\infty}\frac{1}{h^3}e^{-\frac{\varepsilon}{kT}}\frac{dxdydzdpdqdr}{h^3}\right)\right]. \end{aligned} \tag{7.1.19}$$

Upon evaluating the integral one finds:

$$F = Nm_0c^2 - kNT\log\left[\frac{eV}{Nh^3}(2\pi m_0 kT)^{3/2}\right], \tag{7.1.20}$$

where V is the total volume of the gas, so that it follows that:

$$S = kN\log\left[\frac{e^{5/2}V}{Nh^3}(2\pi m_0 kT)^{3/2}\right], \tag{7.1.21}$$

and

$$E = Nm_0c^2 + \frac{3}{2}kNT. \tag{7.1.22}$$

At the end of his book "Warmestrahlung", PLANCK showed how one can deduce the "chemical constant" involved in equilibrium of a gas with its condensed (liquid) phase. Measurements have verified PLANCK's method.

So far we have made use of neither Relativity nor our ideas relating dynamics with waves. We shall now examine how these two aspects are to be introduced into the above formulas.

7.2. The new conception of gas equilibrium

If moving atoms of a gas are accompanied by waves, the container must then contain a pattern of standing waves. We are naturally drawn to consider how within the notions of black body radiation developed by M. JEANS, these phase waves forming a standing pattern (that is, with respect to a container) as the only stable situation, can be incorporated into the study of thermodynamic equilibrium. This is somehow an analogue to a Bohr atom, for which stable trajectories are defined by stability conditions such that unstable waves would be regarded as unphysical.

One can question how there can exist a stable wave formation in view of the fact that atoms of a gas are in chaotic motion due to constant collisions with each other. To begin, one can respond that thanks to the uncoordinated character of atomic motion, the number of atoms deflected from their initial motion during a time interval dt by cause of collisions is exactly compensated by the number redirected into this very same direction; all transpires as if the original atoms traversed a container without any deflections at all. Moreover, during free travel, a phase wave can travel many times the length of a container, even of large dimensions; if, for example, the mean velocity of an atom is 10^5 cm./sec. and the mean free travel 10^{-5} cm., the mean velocity of the phase wave would be $c^2/v = 9\times10^{15}$ cm./sec., and during the time interval 10^{-10} sec. necessary on average for collision free travel, this atom traverses 9×10^5 cm., or 9 kilometres. It seems quite possible, therefore, to imagine stationary phase waves in a gas of massive atoms at equilibrium.

To better understand the nature of the modifications we shall try to impose on statistical mechanics, let us consider at the start the simple case of molecules that move along the line AB of length l, that are reflected at A and B. The initial distribution of velocities is to be random. The probability that a molecule is found in an element dx of AB is therefore dx/l. According to classical notions, one can take the probability that the velocity is between v and $v+dv$ as being proportional to dv; therefore, if one considers phase space spanned by x and v, all elements $dxdv$ are equally probable. The situation is very different, however, when the stability conditions discussed above are taken into consideration. If the velocities are low enough to justify ignoring relativistic terms, the wave length of a wave moving with an atom whose velocity is v, would be:

$$\lambda = \frac{c/\beta}{m_0c^2/h} = \frac{h}{m_0v}, \tag{7.2.1}$$

and the resonance condition is:

(7.2.2) $$l = n\lambda = n\frac{h}{m_0 v}, \quad (n \text{ integer}).$$

Let $h/m_0 l = v_0$; then:

(7.2.3) $$v = nv_0.$$

The velocity therfore is restricted to multiples of v_0.

The variation δn of the whole number n corresponding to a variation δv of the velocity gives the number of states compatible with existence of stationary phase waves. From this one sees:

(7.2.4) $$\delta n = \frac{m_0 l}{h}\delta v.$$

All transpires as if, in each element of phase space $\delta x \delta v$, there corresponds $m_0 \delta x \delta v/h$ possible states, which is the classical expression divided by h. Numerical evaluation shows that even for extremely small values of δv on the scale of experiments, there corresponds a large interval δn; thus every small rectangle in phase space represents an enormous number of possible values of v. On can take it that in general the quantity $m_0 \delta x \delta v/h$ can be handled as an infinitesimal. But, in principle, the distribution of representative points is the same as that imagined in statistical mechanics; it is taken to be discontinuous, and by a mechanism which is yet to be fully determined, the motion of atoms for which there is no stable wave configuration are automatically excluded.

Let us now consider a gas in three dimensions. The distribution of phase waves in a container is fully analogous to that used in the usual analysis of black body radiation. One may, just as M. JEANS did in this case, calculate the number of stationary waves for which the frequency is between ν and $\nu + \delta\nu$. One finds in this case, distinguishing between group velocity U and phase velocity V , the following expression:

(7.2.5) $$n_\nu \delta\nu = \gamma\frac{4\pi}{UV^2}\nu^2\delta\nu,$$

where γ equals 1 for longitudinal waves and 2 for transverse waves. Eq. (7.2.5) must not be misinterpreted; not all values of v are present in every situation, nevertheless it is possible for the purposes of calculation to regard it as a differential, as in general in every small interval there is an enormous number of admissible values of v.

Here the occasion has arrived to use the theorem demonstrated in §1.2. An atom of velocity $v = \beta c$, corresponds to a wave having phase velocity $V = c/\beta$, with the group velocity $U = \beta c$ and frequency $\nu = (1/h)(m_0 c^2/\sqrt{1-\beta^2})$. If w designates the kinetic energy, one finds according to the relativistic formulas:

(7.2.6) $$h\nu = \frac{m_0 c^2}{\sqrt{1-\beta^2}} = m_0 c^2 + w = m_0 c^2(1+\alpha), \quad (\alpha = w/m_0 c^2).$$

From which it follows:

$$n_\omega d\omega = \gamma \frac{4\pi}{UV^2} \nu^2 d\nu = \gamma \frac{4\pi}{h^3} m_0 c^2 (1+\alpha) \sqrt{\alpha(\alpha+2)} d\omega. \tag{7.2.7}$$

Calling on the canonical distribution mentioned above, gives the number of atoms in the volume element $dxdydz$ with kinetic energy between ω and $\omega + d\omega$:

$$C\gamma \frac{4\pi}{h^3} m_0 c^2 (1+\alpha) \sqrt{\alpha(\alpha+2)} e^{-\frac{\omega}{kT}} d\omega dxdydz. \tag{7.2.8}$$

For atoms, phase waves by reason of symmetry are analogous to longitudinal waves, so we take $\gamma = 1$. Moreover, for these atoms (except for a negligible small number at normal temperatures), their proper or rest energy $m_0 c^2$ is substantially greater than their kinetic energy. Thus, we may take $1+\alpha$ to be very close to 1 and therefore:

$$C \frac{4\pi}{h^3} m_0^{\frac{3}{2}} \sqrt{2\omega} e^{-\frac{\omega}{kT}} d\omega dxdydz = Ce^{-\frac{\omega}{kT}} \int_\omega^{\omega + d\omega} \frac{dxdydzdpdqdr}{h^3}. \tag{7.2.9}$$

Obviously, this method shows that the number of possible molecular states in phase space is not the infinitesimal element itself but this element divided by h^3. This verifies PLANCK's hypothesis and thereby a result obtained above. We note that the values of the velocities that lead to this result are those from JEAN's formula.[1]

7.3. The photon gas

If light is regarded as comprising photons, black body radiation can be considered as a gas in equilibrium with matter similar to a saturated vapour in equilibrium with its condensed phase. We have already shown in CHAPTER 3 that this idea leads to an exact expression for radiation pressure.

Let us apply Eq. (7.2.8) to this gas. Here $\gamma = 2$ by reason of symmetry of units as emphasised in CHAPTER 4. In so far as α is large with respect to 1, (except for a number of atoms negligible at usual temperatures) , both $\alpha+1$ and $\alpha+2$ may be replaced with α. Thus, one gets for the number of photons per unit volume with energy between $h\nu$ and $h(\nu + d\nu)$:

$$C \frac{8\pi}{c^3} \nu^2 e^{-\frac{h\nu}{kT}} d\nu dxdydz, \tag{7.3.1}$$

for energy density corresponding to these frequencies:

$$u_\nu d\nu = C \frac{8\pi h}{c^3} \nu^3 e^{-\frac{h\nu}{kT}} d\nu. \tag{7.3.2}$$

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[1] On this matter see: SACKUR, O. *Ann. d. Phys.*, **36**, 958 (1911), and **40**, 67 (1913); TETRODE, H., *Phys. Zeitschr.*, **14**, 212 (1913); *Ann. d. Phys.*, **38**, 434 (1912); KEESOM, W. H., *Phys. Zeitschr.*, **15**, 695 (1914); STERN, O., *Phys. Zeitschr.*, **14**, 629 (1913); BRODY, E., *Zeitschr. f. Phys.*, **16**, 79 (1921).

The constant can be seen to have the value 1 by arguments presented in my article entitled “Quanta de lumière et rayonnement noir” in *Journal de Physique,* 1922.

Unfortunately the law obtained in this way is WIEN’s Law, i.e., only the first term in a series of the exact law found by M. PLANCK. This should not surprise us, for by supposing that moving photons are completely independent of each other, we necessarily come to a result for which the exponent is that found in MAXWELL’s distribution.

We know, incidentally, that a continuous distribution of radiant energy in space leads to RAYLEIGH’s Law as JEANS has shown. But, PLANCK’s Law goes to the expressions proposed by MM. WIEN and RAYLEIGH as limits whenever $h\nu/kT$ is very large or small respectively. To get PLANCK’s Law *a new hypothesis* is needed, which, without abandoning the notion of the existence of photons, will permit us to explain why the classical formulas are valid in certain domains. This hypothesis can formulated thusly:

If two or more photons have phase waves that exactly coincide, then since they are carried by the same wave their motion can not be considered independent and these photons must be treated as identical when calculating probabilities. Motion of these photons “as a wave” exhibits a sort of coherence of inexplicable origin, but which probably is such that out-of-phase motion is rendered unstable.

This coherence hypothesis allows to reproduce in its entirety a demonstration of MAXWELL’s Law. In so far as we can no longer take each photon as an independent “object” of the theory, it is the elementary stationary phase waves that play this role. What shall we call such an elementary stationary wave? A stationary wave may be regarded as a superposition of two waves of the form:

$$\begin{matrix}\sin\\ \cos\end{matrix}\left[2\pi\left(\nu t \mp \frac{x}{\lambda} + \varphi_0\right)\right], \tag{7.3.3}$$

where φ_0 can take on any valuer between 0 and 1. By specifying a value for ν and φ_0, a particular elementary standing wave is defined. Consider now for a particular value ofφ_0 all the permissable values of ν in a small interval $d\nu$. Each elementary wave can transport 0, 1, 2... photons and, because the canonical distribution law may be applied to these waves, one gets for the number of corresponding photons:

$$N_\nu d\nu = n_\nu \frac{\sum_1^\infty p e^{-p\frac{h\nu}{kT}}}{\sum_0^\infty e^{-p\frac{h\nu}{kT}}}. \tag{7.3.4}$$

If φ_0 takes on other values, one gets other stable states; and, by superposing several of these stable states, that correspond to one and the same elementary wave, one gets yet a further stable state. Therefrom we see that the number of photons for which the energy is between ν and $\nu + d\nu$ is:

$$N_\nu d\nu = A\gamma\frac{4\pi}{h^3}m_0c^2(1+\alpha)\sqrt{\alpha(\alpha+2)}d\omega\sum_1^\infty e^{-p\frac{m_0c^2+\omega}{kT}}, \tag{7.3.5}$$

per unit volume. A can be a function of temperature.

For a gas, in the usual sense of the word, m_0 is so large that one may neglect all terms but the first in the series. For this case, one recovers Eq. (7.2.8).

For a photon gas, however, one finds:

$$N_\nu d\nu = A\frac{8\pi}{c^3}\nu^2\sum_1^\infty e^{-p\frac{h\nu}{kT}}d\nu, \tag{7.3.6}$$

and, therefrom, the energy density:

$$u_\nu d\nu = A\frac{8\pi h}{c^3}\nu^3\sum_1^\infty e^{-p\frac{h\nu}{kT}}d\nu. \tag{7.3.7}$$

This is actually PLANCK's formula. But, it must be noted that in this case $A = 1$. First of all, it is certainly true here that A is not a function of temperature. In fact, total radiation energy per unit volume is:

$$u = \int_0^{+\infty} u_\nu d\nu = A\frac{48\pi h}{c^3}\left(\frac{kT}{h}\right)^4\sum_1^\infty\frac{1}{p^4}, \tag{7.3.8}$$

and total entropy is given by:

$$\begin{aligned} dS = \frac{1}{T}\left[d(uV) + PdV\right] &= V\frac{du}{T} + (u+P)\frac{dV}{T}, \\ &= \frac{V}{T}\frac{du}{dT}dT + \frac{4}{3}u\frac{dV}{T}. \end{aligned} \tag{7.3.9}$$

where V is total volume; and, because $u = f(T)$ and $P = (u - dS)/3$, this expression is an exact differential where the integrability condition can be written:

$$\frac{1}{T}\frac{du}{DT} = \frac{4}{3}\frac{1}{T}\frac{du}{dT} - \frac{4}{3}\frac{u}{T^2}, \text{ or } 4\frac{u}{T} = \frac{du}{dT}, \quad u = \alpha T^4. \tag{7.3.10}$$

This is the classical form of STEFAN's Law, which leads to setting $A = C$. The reasoning used above gives us the values of the entropy:

$$S = A\frac{64\pi}{c^3h^3}k^4T^2V\sum_1^\infty\frac{1}{p^4}, \tag{7.3.11}$$

and free energy:

$$F = U - TS = -A\frac{16\pi}{c^3h^3}k^4T^4V\sum_1^\infty \frac{1}{p^4}. \tag{7.3.12}$$

It remains only to determine the value of A. If it turns out that it can be shown to be equal to 1, we shall get PLANCK's formulas.

As remarked above, if one neglects terms where $p > 1$, the matter is such that, the distribution of photons obeys the simple canonical law:

$$A\frac{8\pi}{c^3}\nu^2 e^{-\frac{h\nu}{kT}}d\nu, \tag{7.3.13}$$

and one can calculate the free energy using PLANCK's method for an ordinary gas, so that identifying the result with expression above, one sees that: $A = 1$.

In the general case, one must use a less direct method. Consider the p-th term in PLANCK's series:

$$n_{\nu p}d\nu = A\frac{8\pi}{c^3}h\nu^3 e^{-p\frac{h\nu}{kT}}d\nu. \tag{7.3.14}$$

One may this as:

$$A\frac{8\pi}{c^3 p}\nu^2 e^{-p\frac{h\nu}{kT}}d\nu \cdot p \cdot h\nu, \tag{7.3.15}$$

which admits the claim:

Black body radiation can be considered to be a mixture of infinitely many gases each characterised by one whole number p and possessing the property that, the number of states of a gaseous totality located in the volume dxdydz and having energy between phν *and ph*$(\nu + d\nu)$ *equals* $(8\pi/c^3p)\nu^2 d\nu dxdydz$. From this, one can calculate free energy using the method in §7.1. One gets:

$$\begin{aligned} F = \sum_1^\infty F_p &= -kT\sum_1^\infty \log\left[\frac{1}{n_p!}\left(V\int_0^\infty \frac{8\pi}{c^3p}\nu^2 e^{-p\frac{h\nu}{kT}}d\nu\right)^{n_p}\right], \\ &= -kT\sum_1^\infty \log\left[\frac{e}{n_p}V\int_0^\infty \frac{8\pi}{c^3p}\nu^2 e^{-p\frac{h\nu}{kT}}d\nu\right], \end{aligned} \tag{7.3.16}$$

where:

$$n_p = V\int_0^\infty A\frac{8\pi}{pc^3}\nu^2 e^{-p\frac{h\nu}{kT}}d\nu = A\frac{16\pi}{c^3}\frac{k^3T^3}{h^3}\frac{1}{p^4}V. \tag{7.3.17}$$

So, finally:

$$F = -A\frac{16\pi}{c^3h^3}k^4T^4\log\left(\frac{e}{A}\right)\sum_1 \frac{1}{p^4}V, \tag{7.3.18}$$

and, by identification with the expression above, it follows:

$$\log\left(\frac{e}{A}\right) = 1, \quad A = 1, \tag{7.3.19}$$

which is what we want to show.

The coherence hypothesis adopted above has lead us to good results and, moreover, we avoided foundering on either of the laws of RAYLEIGH or WIEN. The study of its fluctuations has provided us a new proof of the importance of black body radiation.

7.4. Energy fluctuations in black body radiation[2]

If energy parcels of value q are distributed in very large quantitates in a given space and if their positions vary ceaselessly and randomly, a volume element normally containing $\bar{n}$ parcels, has energy $\bar{E} = \bar{n}q$. But, the actual value of n varies considerably from $\bar{n}$, which, from a certain theorem of probability theory satisfies $\overline{(n-\bar{n})^2} = \bar{n}$, so that the mean square fluctuation of energy would be:

$$\overline{\varepsilon^2} = \overline{(n-\bar{n})^2} q^2 = \bar{n}q^2 = \bar{E}q. \tag{7.4.1}$$

On the other hand, one knows that energy fluctuations of black body radiation in a volume V are given by a law of statistical thermodynamics, namely:

$$\overline{\varepsilon^2} = kT^2 V \frac{d(u_\nu d\nu)}{dT}, \tag{7.4.2}$$

for the interval ν to $\nu + d\nu$. Now, using RAYLEIGH's Law, one gets:

$$u_\nu = \frac{8\pi k}{c^3}\nu^2 T, \quad \overline{\varepsilon^2} = \frac{c^3}{8\pi\nu^2 d\nu} \cdot \frac{(Vu_\nu d\nu)^2}{V}, \tag{7.4.3}$$

and this result, as might be expected, corresponds to that obtained considering interference in electrodynamics.

If, on the other hand, one takes WIEN's Law, which corresponds to the hypothesis that radiation is comprised of independent photons, one gets:

$$\overline{\varepsilon^2} = kT^2 V \frac{d}{dT}\left(\frac{8\pi h}{c^3}\nu^3 e^{-\frac{h\nu}{kT}} d\nu\right) = (u_\nu V d\nu)h\nu, \tag{7.4.4}$$

which again leads directly to $\overline{\varepsilon^2} = \bar{E}h\nu$.

Finally, for the truly realistic case, i.e., using PLANCK's Law, one finds:

$$\overline{\varepsilon^2} = (u_\nu V d\nu)h\nu + \frac{c^3}{8\pi\nu^2 d\nu} \cdot \frac{(Vu_\nu d\nu)^2}{V}, \tag{7.4.5}$$

[2] EINSTEIN A., Die Theorie der Schwartzen Strahlung und die Quanten, *Proceedings Solvay Conference,* p, 419; LORENTZ, H.-A., Les Théories statistiques en theromdynamique, *Reunion Conférences de* M. H.-A. LORENTZ *au Collège de France,* (Teuner, Leipzig, 1916) pp. 70 and 114.

$\overline{\varepsilon^2}$ therefore appears to be the sum of a term for which radiations would be independent parcels, $h\nu$, and a term for which it should be purely undulatory.

On the other hand, the notion that collections of photons comprise "waves" leads us to write PLANCK's Law:

$$u_\nu d\nu = \sum_1^\infty \frac{8\pi h}{c^3}\nu^3 e^{-p\frac{h\nu}{kT}}\,d\nu = \sum_1^\infty n_{p,\nu} p h\nu d\nu, \tag{7.4.6}$$

and, by applying the formula $\overline{\varepsilon^2} = \overline{n}q^2$ to each type of grouping, one gets:

$$\overline{\varepsilon^2} = \sum_1^\infty n_{p,\nu} d\nu (ph\nu)^2. \tag{7.4.7}$$

Naturally, this expression is at root identical to EINSTEIN's, only its written form is different. But, it is interesting in that it brings us to say: *One can correctly account for fluctuations in black body radiation without reference to interference phenomena by taking it that this radiation, as a collection of photons, has a coherent phase wave.*

It thus appears virtually certain that every effort to reconcile discontinuity of radiant energy and interference will involve the hypothesis of coherence mentioned above.

Appendix to CHAPTER 5: Light quanta

We proposed considering photons of frequency ν as small parcels of energy characterised by a very small proper mass m_0 and always in motion at a velocity very nearly identical to the speed of light c, in such a way that there is among these variables the relationship:

$$h\nu = \frac{m_0 c^2}{\sqrt{1-\beta^2}}, \tag{7.4.8}$$

from which one deduces:

$$\beta = \sqrt{1 - \left(\frac{m_0 c^2}{h\nu}\right)^2}. \tag{7.4.9}$$

This point of view led us to remarkable compatibilities between the Doppler Effect and radiation pressure.

Unfortunately, it is also subject to a perplexing difficulty: for decreasing frequencies ν, the velocity βc of energy transport also gets lower, such that when $h\nu = m_0 c^2$ it vanishes or becomes imaginary (?). This is more difficult to accept than, that in the low frequency domain one should, in accord with the old theories, also assign the velocity c to radiant energy.

This objection is very interesting because it brings attention to the issue of passage from the purely high frequency corpuscular regime to the purely low frequency undulatory regime. We have shown in CHAPTER 7 that corpuscular notions lead to WIEN's Law, as is well known, while undulatory ideas lead to RAYLEIGH's Law. The passage from one to the other of these laws, it seems to me, must be closely related to the above objection.[4]

I shall, by means of an example with the hope of providing a resolution of this difficulty, develop a notion suggested by the above considerations.

[4] It may be of historical interest, that the remaining material in this appendix was omitted in the German translation. -A.F.K.

In CHAPTER 7 I have shown how passage from WIEN's to RAYLEIGH's Law is explicable in terms of a coherent phase wave for an ensemble of photons. I have emphasised the similarity between such a Phase wave with a large number of photons and a classical wave. However, this similarity is sullied by the fact that each photon represents a finite mass m_0 although the classical theory of electromagnetism attributes no mass at all to light. The frequency of a phase wave containing multiple photons is given by:

$$h\nu = \frac{\mu_0 c^2}{\sqrt{1-\beta^2}}, \tag{7.4.10}$$

where μ_0 is the proper mass of each photon, which seems necessary so as to be able to compute absorption and emission of finite quantities of energy, $h\nu$. But we may, perhaps, suppose that the mass of photons allied with the same phase wave differs from the mass of an isolated photon. One might take it that photon mass is a function of the number of photons, p, allied with a phase wave:

$$\mu_0 = f(p), \quad \text{with} \quad f(1) = m. \tag{7.4.11}$$

The necessity to return to classical formulas for low frequencies leads us to suppose that $f(p)$ tends to 0 as $p \to \infty$. Thus, the ensemble velocity would be given by:

$$\beta c = c\sqrt{1 - \left(\frac{f(p)c^2}{h\nu}\right)^2}. \tag{7.4.12}$$

For very high frequencies, p would always equal 1 giving for isolated photons WIEN's Law for black body radiation and the formula: $\beta = \sqrt{1-(m_0c^2/h\nu)^2}$ for the energy transport velocity. For low frequencies, p is always very large, photons are found always in numerous ensembles allied with the same phase wave; black body radiation follows RAYLEIGH's Law, and the transport velocity goes to c as $\nu \to 0$.

This hypothesis undermines the simplicity of the concept of "photon", but this simply can not be maintained and still reconcile electrodynamics with discontinuous photoelectric phenomena. Introducing $f(p)$, it seems to me, reconciles photon population idiosyncrasies with classical wave notions.

In any case, the true structure of radiant energy remains very mysterious.

Summary and conclusions

The rapid development of Physics since the XVII*th* century, in particlar the development of Dynamics and Optics, as we have shown, anticipates the problem of understanding quanta as a sort of parallel manifestation of corpuscles and waves; then, we recalled how the notion of the existence of quanta insessently engages the attention of researchers in the XX*th* century.

In CHAPTER 1, we introduced as a fundamental postulate the existence of a periodic phenomena allied with each parcel of energy with a proper mass given by the Planck-Einstein relationship. Relativity theory revealed the need to associate uniform motion with propagation of a certain "phase wave" which we placed in a MINKOWSKI space setting.

Returning, in CHAPTER 2, to this same question in the general case of a charged particle in variable motion under the influence of an electromagnetic field, we showed that, following our ideas, MAUPERTUIS' principle of least action and the principle of concordance of phase due to FERMAT can be two aspects of the same law; which led us to propose that an extention of the quantum relation to the velocity of a phase wave in an electromagnetic field. Indeed, the idea that motion of a material point always cloaks propagation of a wave, needs to be studied and extended, but if it should be formulated satisfactorily, it represents a truly beautiful and rational synthesis.

The most important consequences are presented in CHAPTER 3. Having recalled the laws governing stability of trajectories as quantified by numerous recent works, we have shown how they may be interpreted as expressions of phase wave resonance along closed or semi-closed trajectories. We believe that this is the first physical explanation of the Bohr-Sommerfeld orbital stability conditions.

Difficulties arising from simultaneous motion of interacting charges were studied in CHAPTER 4, in particlar for the case of circular orbital motion of an electron and proton in an hydrogen atom.

In CHAPTER 5, guided by preceeding results, we examined the possibility of representing a concentration of energy about certain singularities and we showed what profound harmony appears to exist between the opposing viewpoints of NEWTON and FRESNEL which are revealed by the identity of various forecasts. Electrodynamics can not be maintained in its present form, but reformulation will be a very difficult task for which we suggested a qualitative theory of interferences.

In CHAPTER 6 we reviewed various theories of scattering of X and γ-rays by a amorphous materials with emphasis on the theory of MM. P. DEBYE and A.-H. COMPTON, which render, it seems, existence of photons as a tangible fact.

Finally, in CHAPTER 7 we introduced phase waves into Statistical Mechanics and in so doing recovered both the size of the elemental extention of phase space, as determined by PLANCK, as well as the black body law, MAXWELL's Law for a photon gas, given a certain coherence of their motion, a coherence also of utility in the study of energy fluctuations.

Briefly, I have developed new ideas able perhaps to hasten the synthesis necessary to unify, from the start, the two opposing, physical domains of radiation, based on two opposing conceptions: corpuscles and waves. I have forecast that the principles of the dynamics of material points, when one recognises the correct analysis, are doubtlessly expressible as phase concordance and I did my best to find resolution of several mysteries in the theory of the quanta. In the course of this work I came upon several interesting conclusions giving hope that these ideas might in further development give conclusive results. First, however, a reformulation of electrodynamics, which is in accord with relativity of course, and which accommodates discontinuous radiant energy and phase waves leaving the Maxwell-Lorentz formulation as a statistical approximation well able to account accurately for a large number of phenomena, must be found.

I have left the definitions of phase waves and the periodic phenomena for which such waves are a realization, as well as the notion of a photon, deliberately vague. The present theory is, therefore, to be considered rather tentative as Physics and not an established doctrine.

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