# THE MULTI-BODY PROBLEM AND RESONANCE
# IN QUANTUM MECHANICS

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ABSTRACT. This paper presents an attempt to treat the multi-body problem in Quantum Mechanics. For this purpose we examine carefully a resonance phenomenon characteristic of quantum multi-body interactions which is interrelated with BOSE-EINSTEIN statistics and PAULI's Exclusion Principle.

To date, Quantum Mechanics (QM) [For HEISENBERG, QM is 'matrix mechanics.' — trans.] has been applied only to systems consisting of a single, mobile, massive body. This limitation is a consequence of the mathematical difficulties encountered computing individual amplitudes. Recently great progress has been made by SCHRÖDINGER,[1] on the basis of DE BROGLIE's wave theory[2], which provides a substantially more mathematically convenient approach to QM. Just as the formal similarity of HAMILTON's optical analogy to classical mechanics set in multidimensional spaces led to effective mathematical methods, so now SCHRÖDINGER's ideas also lead to effective mathematical techniques, also in multidimensional spaces, to attack QM problems. For a system with $f$ degrees of freedom, SCHRÖDINGER substitutes for the QM problem, an eigenvalue problem in a space of $f$ dimensions; where the energies of the stationary states are the eigenvalues, and where the coefficients of series expansions in terms of eigenfunctions (once they have been found) can be obtained simply by integration. There is, however, a difference in the physical interpretation between SCHRÖDINGER's approach and ours. In view of the mathematical equivalence of these two approaches, one could consider the issue of the underlying physical processes as a variation of our question on the usefulness of such considerations; at least when the purpose is not to extend QM on the basis of the resulting imagery. To mention a specific example: to treat the multi-body problem, while we strive to obtain a viewpoint in accord with QM, SCHRÖDINGER's presentation techniques[3] seem to evoke considerable modifications in the usual equations. I myself, with regard to interpretation issues, believe that there are cases for which the wave picture is more forced than one simply envisioning particles moving around in a space in which a non-classical kinematics obtains. One recalls, for example, the image of a spinning electron. So far as I can see, moreover, SCHRÖDINGER's approach is not a consequential wave theory of the sort promoted by DE BROGLIE. The transfer to an $f$-dimensional space and computation of the wave velocity from the particles' mutual potential energy, represents a loan from the stock of concepts pertaining to particulate matter. Even were a serious wave theory of matter

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[1] E. SCHRÖDINGER, *Ann. d. Phys.* **79**, 361, 489, 734 (1926).

[2] L. DE BROGLIE, *Ann. de phys.,* **3** (10), 22 (1925).

[3] Ref. [1], footnote, p. 750.

developed in accord with DE BROGLIE's and EINSTEIN's program, there still would not exist an exhaustive description of atomic processes in terms of our usual space-time conceptions. Despite the ever narrowing, close analogy between light and matter, one might consider that a wave theory of matter provides just as limited a description of the empirical results of atomic physics as the wave theory of light does for optics. In the light of this analogy, it seems to me to be an advantage for QM that it is based on particle ideas for matter; of course this does not mean that their motion is governed by the usual space-time considerations. This situation might have been anticipated, since if particles should be singularities in the metric structure of space, as is the aim of proponents of continuum theories, such would not be a description in terms of our usual space-time concepts—unless, one were to consider a space deviating from the usual Euclidean space to be a "normal" space.

In any case, I do not wish to discuss further these difficult problems of physical interpretation, which, at the current state of theory, cannot be resolved. We are happy to note that the SCHRÖDINGER technique and QM mutually enrich each other, both mathematically and in terms of imaginable content and analogies, so as to enable ever deeper penetration into the physical essence of very small scale processes.

The goal of this research is a method for the QM treatment of multi-body systems. Such appears to be faced with difficult obstacles, namely: the elements of DE BROGLIE's wave theory that accommodates BOSE-EINSTEIN statistics[4], seem to have no analogue in QM; auxiliary conditions, such as PAULI's Exclusion Principle[5], also seems not to fit in this formulation of QM. One might suspect that QM fails for the situation involving equivalent orbits. Finally, let us recall a difficulty in the quantitative treatment of spectra: the separation between singlet and triplet lines in the spectrum of alkaline earth metals and Helium are an order of magnitude too great to be accounted for by the interaction of spinning electrons.

The goal of the following research is an exact analysis of the conclusions one can draw from a serious application of QM to the multi-body problem. To announce our results in advance: we shall see that the problems mentioned above are resolved, such that a natural interrelationship between BOSE-EINSTEIN statistics and QM can be recognized.

§1. The simplest imaginable multi-body system consists of two oscillators. As is well known, such a system can always be decoupled if the energy of interaction is a quadratic function of the coordinates. No one doubts that this system can be handled with QM without further assumptions. Moreover, this example contains all the features characteristic of the quantum theoretical multi-body problem, so that with its analysis one finds all the techniques needed later for the clarification of atomic spectra. This coupled-oscillator problem exhibits yet another advantage: it is that there are virtually no differences between its classical theory, the old quantum theory [BOHR's theory —trans.] and QM; for each quantum result there is an classical mechanical analogue. However, the analysis of more complex problems does not always accommodate such classical analogues.

It is a characteristic feature of atomic systems, that the component subsystems of which they are comprised, namely the electrons, are identical and subject to identical forces. In order to invest this feature into our problem, we assume the HAMILTONian to be:

$$(1) \qquad H = \frac{1}{2m}p_1^2 + \frac{m}{2}\omega^2 q_1^2 + \frac{1}{2m}p_2^2 + \frac{m}{2}\omega^2 q_2^2 + m\lambda q_1 q_2;$$

[4]N. S. BOSE, Z. f. Phys., **26**, 178 (1924). The relevant application of these statistics for particles is found in: A. EINSTEIN, *Sitzungsber. d. preuß. Akad. d. Wiss.*, p. 261 (1924) & pp. 3, 8 (1925).

[5]W. PAULI, Z. f. Phys., **31**, 765 (1925).

i.e., the frequencies and masses of the coupled oscillators are taken to be identical. In Eq. (1), $q_1$, $q_2$ denote the coordinates, $p_1$, $p_2$ the momenta, $m$ and $\omega$ the mass and frequency respectively and $\lambda$ the interaction constant. With help of the well known transformations:

$$(2) \qquad q_1' = \frac{1}{\sqrt{2}}(q_1 + q_2), \quad q_2' = \frac{1}{\sqrt{2}}(q_1 - q_2),$$

Eq. (1) is transformed into:

$$(3) \qquad H = \frac{1}{2m}p_1'^2 + \frac{m}{2}\omega_1'^2 + \frac{1}{2m}p_2'^2 + \frac{m}{2}\omega_2'^2 q_2'^2,$$

where

$$(4) \qquad \omega_1'^2 = \omega^2 + \lambda, \quad \omega_2'^2 = \omega^2 - \lambda.$$

$H$ separates into the sum of two oscillators, such that each corresponds to a "principle resonance frequency" or mode. When only the first mode, $q_1'$, is excited, then both masses oscillate in phase, and when only $q_2'$ is excited, out of phase, in other words, with a phase difference of $\pi$. (See Fig. 1.)

Fig. 1

The energies [according to QM] for the combined system are then give by the equation:

$$(5) \qquad H_{n_1', n_2'} = \frac{\omega_1' h}{2\pi}\left(n_1' + \frac{1}{2}\right) + \frac{w_2' h}{2\pi}\left(n_2' + \frac{1}{2}\right),$$

where $n_1'$ and $n_2'$ must be whole numbers, i.e., integers. Taking as a symbol for each term '$n_1' n_2'$,' one finds the term schematic (i.e., energy level diagram) shown in Fig. 2, where the difference between the $+$ and $\circ$ is to be ignored for the moment.

Now we wish to explore the various transformation processes. In order to make as close an analogy as possible with an atomic system, we assume that both oscillators are made from identical masses and constrained to oscillate along a single line. The electric dipole moment is then given in large measure by $q_1 + q_2$. From these conditions it follows as once, as is evident also on Fig. 1, that only the first oscillation mode gives rise to an electric moment. Thus, in this approximation, there can be only increments of 1 for '$n_1'$', that is, on Fig. 2, only transitions that are vertical, no crossovers. For a dipole moment source, there is radiation only for the first term; quadratic and higher poles yield only insignificant contributions. Such higher terms for the radiation are determined by homogeneous symmetrical functions of the second order in $q_1$ and $q_2$ and their time derivatives. When $q_1$ and $q_2$ are replaced us-

Fig. 2

ing Eq. (2) by $q_1'$ and $q_2'$, the terms that arise are homogeneous functions of the second and third order in $q_1'$ and $q_2'$, and which, by cause of the symmetry in $q_1$ and $q_2$, allow the coordinates $q_2'$ along with their time derivatives, to be expressed only in an even number of factors. That means, that by all higher approximations for radiation, $n_2'$ can vary only by an integer. Thus, the transition schematic, Fig. 2, can be decomposed into two partial systems ($+$ and $\circ$), within which all transitions are strictly confined. Not even excitation by collision, which can be considered stimulation by a combination of various multipoles, can cause transitions between these partial systems. This exclusion is, however, strictly dependant on the absolute identity of the two coupled oscillators. Whenever even a small
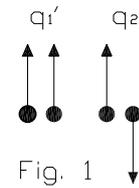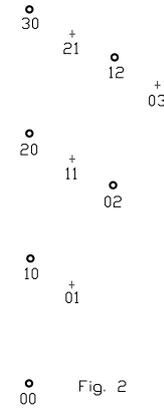
difference in mass or frequency is involved, transitions between the subsystems become possible, for which the intensity is proportional to the size of such differences.

Let us return to consideration of identical oscillators; in this case, in the QM treatment there appears a characteristic indeterminacy. The question then becomes: in reality are both systems present, or only one or the other partial subsystem? Actually one asks of a quantum solution to a problem only that the resulting transition schematic be "closed," i.e., that all lines, and only those lines, appear that actually occur in nature; in other words, that all impossible transitions are assigned a vanishing probability. Therefore, we may say that both the $+$ and the $\circ$ systems, and their combination, are suitable quantum solutions for this problem. The quantum indeterminacy seems to me to be the significant aspect of the solution to this problem. It provides the flexibility permitting both BOSE-EINSTEIN statistics and PAULI's exclusion principle to be accommodated within QM. Before detailed examination of this matter, I would like to reemphasize that coupled systems in QM always exhibit the features seen in this problem.

§2. In classical mechanics one finds that two periodic oscillating systems can come into resonance if the frequencies of the separate systems are independent of the energy and are approximately equal. Resonance in this sense occurs only among harmonic oscillators. In QM—in accord with general laboratory observations—two atomic systems resonate when the absorption frequency of one system coincides with the emission frequency of the other; because of the essential linearity of quantum equations, resonance there is a much more common phenomenon than it is in classical theory.

For detailed analysis, let us consider two identical systems, $a$ and $b$, each having $f$ degrees of freedom and coupled with a symmetric interaction energy of, $\lambda H^1$. The individual systems must not be degenerate; the energies of their stationary states are given by $H_n^a$ and $H_m^b$.

To begin, let us consider the sum of the two systems while ignoring the interaction, so that the combined energy for the stationary state "$m, n$" is then:

$$(6) \qquad H_{n,m} = H_n^a + H_m^b.$$

This total system suffers then a characteristic degeneration in its resonance because:

$$(7) \qquad H_{n,m} = H_{m,n},$$

i.e., each eigenvalue is doubled, with the exception of those for which $n = m$. In other words, there is always a degeneracy-resonance if the two systems start out in different states; in this case an exchange of the two systems yields the same energy; only when the component systems are in the same state, does the resonance (or the degeneracy) not occur. Fig. 3 is an example of a transition schematic for this case.

In systems with interaction this degeneracy is lifted; and a regular secular oscillation of energy back and forth between the component systems takes place. Formally, the connection is as follows: the additional energy $\mathbf{W}^1$ of the perturbed systems is to first approximation equal to the time averaged mean value of the unperturbed energy $\mathbf{H}^1$. This mean value in general encompasses terms that correspond to transitions in which systems $a$ and
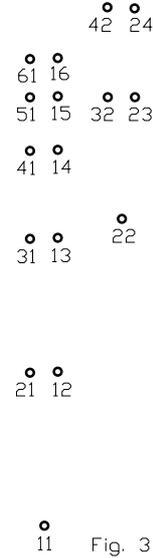
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                    o   o
                    42  24

            o   o
            61  16
            o   o   o   o
            51  15  32  23
            o   o
            41  14
                        o
            o   o       22
            31  13


            o   o
            21  12



            o
            11     Fig. 3
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$b$ exchange places. Thus, one must introduce a canonical transformation such that $\mathbf{W}^1$ becomes a diagonal matrix. This calculation can be found in[6]. The canonical transformation should be:

$$(8) \qquad\qquad \mathbf{W}^1 = \mathbf{S}^{-1}\overline{\mathbf{H}^1}\mathbf{S},$$

$$(9) \qquad\qquad q' = \mathbf{S}^{-1}q\mathbf{S},$$

where $\mathbf{S}$ is a matrix, which like $\overline{H^1}$, has only diagonal elements that correspond to the transitions between equal energy states. For nondegenerate states, the values of the diagonal matrix entries are all equal to '1.' This requires the solution of two linear equations with the unknowns, $S_{n,m}$ and $S_{m,n}$, :

$$(10) \qquad \begin{bmatrix} W^1 - H^1(nm, nm) & H^1(nm, mn) \\ -H^1(mn, nm) & W^1 - H^1(mn, mn) \end{bmatrix} \begin{bmatrix} S_{nm} \\ S_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In view of the symmetry of $H^1$ with respect to the systems $a$ and $b$, i.e., $H^1(nm, nm) = H^1(mn, mn)$ and $H^1(nm, mn) = H^1(mn, nm)$, Eq. (10) becomes:

$$(11) \qquad \begin{bmatrix} W^1 - H^1(nm, nm) & H^1(nm, mn) \\ -H^1(nm, mn) & W^1 - H^1(nm, nm) \end{bmatrix} \begin{bmatrix} S_{nm} \\ S_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is useful, to enumerate both solutions for $W^1$, obtained from Eq. (11) by setting the determinate of the $2 \times 2$ matrix in Eq. (11) equal to 0, by $nm$ and $mn$. It must be kept in mind, however, that these numbers correspond to quantum numbers, $n'$ of the main oscillation modes considered in §1, and not to the states of the component systems. Rather, in each state, both component systems execute the same motion (but with different phases). The solutions to Eqs. (11) turn out to be:

$$(12) \qquad \begin{aligned} W^1_{nm} &= H^1(nm, nm) + H^1(nm, mn); \\ S_{nm,nm} &= 1/\sqrt{2}, \quad S_{mn,nm} = 1/\sqrt{2}; \\ W^1_{mn} &= H^1(nm, nm) - H^1(nm, mn); \\ S_{nm,mn} &= 1/\sqrt{2}, \quad S_{mn,mn} = -1/\sqrt{2}. \end{aligned}$$

The immediate effect of an interaction is to make all energies distinct. The term diagram then becomes that depicted in Fig. 4.

The transition spectrum—and this our crucial result—can be seen to be composed of two, absolutely separate, systems ($+$ and $\circ$ on Fig. 4). Emissions are represented by functions of $\mathbf{p}$ and $\mathbf{q}$, which for an exchange of the subsystems do not change value. If these functions are given by $\mathbf{f}$, then their matrix elements after the canonical transformation, Eq. (8) by $\mathbf{S}$:

$$(13) \qquad\qquad \mathbf{f}' = \mathbf{S}^{-1}\mathbf{f}\mathbf{S},$$

that is, by virtue of the symmetry of $\mathbf{f}$:

1. When: $n_1 \neq m$, $n_2 \neq m$, $n \neq m_1$, $n \neq m_2$,

$$(14) \qquad \begin{aligned} f'_{n_1m,n_2m} &= 1/2(f_{n_1m,n_2m} + f_{mn_1,mn_2} + f_{n_1m,mn_2} + f_{mn_1,n_2m}) = \\ &= f_{n_1m,n_2m} + f_{n_1m,mn_2}. \\ f'_{nm_1,nm_2} &= 1/2(f_{nm_1,nm_2} + f_{m_1n,m_2n} - f_{nm_1,m_2n} - f_{m_1n,nm_2}) = \\ &= f_{nm_1,nm_2} - f_{nm_1,m_2n}. \\ f'_{n_1m,nm_2} &= 1/2(f_{n_1m,n_2m} - f_{mn_1,mn_2} + f_{mn_1,n_2m} - f_{n_1m,mn_2}) = 0. \\ f'_{nm_1,m_2n} &= 1/2(f_{m_1n,m_2n} - f_{nm_1,nm_2} + f_{m_1n,nm_2} - f_{nm_1,m_2n}) = 0. \end{aligned}$$

[6]M. BORN, W. HEISENBERG, and P. JORDAN, 'Quantenmechanik II,' Z. f. Phys., **35**, 557 (1926), p. 589.

2. For combinations of states for which the component systems are in "equivalent states"[7]:

(15)
$$f'_{n_1 m, mm} = 1/\sqrt{2}(f_{n_1 m, mm} + f_{mn_1, mm}) = \sqrt{2} f_{n_1 m, mm}.$$
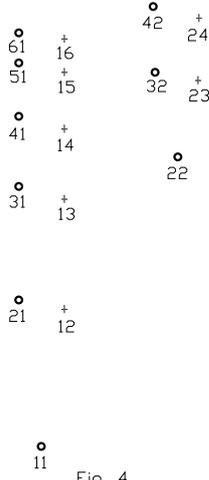$$f'_{mm_1, mm} = 1/\sqrt{2}(f_{m_1 m, mm} - f_{mm_1, mm}) = 0.$$

All combinations between the component subsystems + and ∘ vanish. The line intensities within each subsystem to first approximation are the same as the intensities of the terms for the separate subsystems $a$ and $b$; because amplitudes of the type $f_{n_1 m, m, n_2}$, correspond to simultaneous transitions in both systems, which vanish to first order. However, combinations with those states, that correspond to equivalent states for the systems $a$ and $b$, have an intensity, per Eqs. (15), twice as large as that of a component system. The other row of terms has no such state. The "total intensity" of the combined system, therefore, remains to first approximation unaltered by resonance.

The division into the separate columns of terms cannot be affected by influence from outside. Evidently the very same conditions pertain here as in the special example in §1. The absence of intersystem combinations is a consequence of the absolute identity of the component systems; as soon as these systems are distinguishable, however, intersystem combinations arise. Moreover, the quantum solution has the same indeterminate character as mentioned above; both the system + and ∘ each individually and in combination are valid as a complete solution for the problem.

It may be of interest to consider this resonance phenomenon in terms of SCHRÖDINGER's formulation. To begin, we take as known the original subsystems' normalized eigenfunctions, $\varphi_n^a$ and $\varphi_m^b$, for states with energies $H_n^a$ and $H_m^b$ respectively. Then, for example, the matrix element $n_1 n_2$ of the coordinate $q_k^a$ for the system $a$, is:

$$q_k^a(n_1, n_2) = \int \ldots \int q_k^a \varphi_{n_1}^a \varphi_{n_2}^{a*} dq_1^a \ldots dq_f^a.$$

If one combines the systems, then the energy is given by $H_{n,m} = H_n^a + H_m^b$ and pertains to the state $\varphi_n^a \varphi_m^b$. To the solutions, Eqs. (12), there are eigenfunctions for the perturbed system obtained from the unperturbed system by application of the matrix **S**. Specifically, to the Energies $W_{nm}^1$ and $W_{mn}^1$ correspond, respectively, the following states:

(16)
$$1/\sqrt{2}(\varphi_n^a \varphi_m^b + \varphi_m^a \varphi_n^b),$$
$$1/\sqrt{2}(\varphi_n^a \varphi_m^b - \varphi_m^a \varphi_n^b).$$

The absence of interaction between the component systems $a$ and $b$ follows simply from the fact that, an integral over a function symmetric in $a$ and $b$, or over an operator $f$ of the type:

(17)
$$\frac{1}{2} \int f(\varphi_n^a \varphi_m^b + \varphi_m^a \varphi_n^b)(\varphi_n^a \varphi_m^b - \varphi_m^a \varphi_n^b)^* dq_1^a \ldots dq_f^b,$$

that changes sign when $a$ and $b$ are exchanged, but otherwise retains its value, must equal zero.

§3. As an example of the application of this general theory, we consider briefly here the Helium atom; for comprehensive treatment, I intend to return later.

Fig. 4

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[7]In this paper the expression "equivalent states" refers to the unperturbed system. In perturbed systems, component subsystems always execute the same motion, but with different phases.

3.1. Let us assume that electrons are point charges without magnetic or spin momentum. In this case the essential features of the transition chart are shown in Fig 4. The two systems into which the chart splits, are para- and orthohelium. Transitions between them are essentially impossible. The energy difference between the terms for para- and orthohelium can be attributed to resonance motion caused by mutual Coulomb repulsion. This energy difference is, clearly, of the same order of magnitude as the deviation from the hydrogen-like term that is due to shielding. Moreover, from Fig. 4, it is obvious that the $S1$ term can be present in only one of the component systems. That this is the system for which energy is higher in general, follows from calculations. Transition probabilities within para- and orthohelium are to first approximation identical to those in hydrogen; except that parahelium transitions to the $1S$ term should be roughly twice as frequent.

Should one wish to construct a more or less sensible image or picture of the motion of the electrons in an atom corresponding to the quantum solution, then he must imagine that both electrons are continuously exchanging places with constant period, in analogy to the periodic energy swings seen for the oscillators considered above, where the period of the swinging is given by the difference of parahelium terms from those of orthohelium.

3.2. We are reminded of the COMPTON-UHLENBECK-GOUDSMIT hypothesis[8], to the effect that electrons can be thought of as small magnetic tops, to which we would like to attribute to each a distinct direction to its axis. Then, although little is changed qualitatively in Fig. 4, there arises small interactions between ortho- and parahelium depending on the specific orbits and the strength of the magnetic interaction. In this case, the electrons are no longer strictly identical.

3.3. Let us consider arbitrary directions for these electromagnets; then calculation shows that each term on the chart splits into four terms, corresponding to the statistical weight of the electromagnets, and that the chart retains its form as two completely separated systems, since the electrons are, once again, identical. But the distribution is altered (compare Fig. 5 + and ○.). One system presents a spectrum of terms for which orthohelium is a triplet system and parahelium a singlet, whereas the other system reverses this pattern. Interconnections between ortho- and para-systems occur, as described in §2, with an intensity depending on the interaction between the orbits and electromagnets. On the other hand, there are no interconnections between the + and ○ systems. One can summarize the physical situation as follows: between both electrons there occurs a large electric resonance, which is perturbed by the magnetic interaction so that transitions between ortho- and parahelium do occur. But, with deeper examination it is seen also that finer magnetic resonances occur, which in turn led to a division of the term chart. Calculations supporting this explanation will be published shortly.

§4. For the helium spectrum it is an empirical fact that only one of the systems depicted in Fig. 5 is observed; and, as far as I can see, at least qualitatively, that the other systems do not exist in nature. This fact seems to me—when we assume, that the results derived herein for two component systems can be generalized to an arbitrary number—to imply a connection between the quantum indeterminacy emphasized herein, PAULI's Law and BOSE-EINSTEIN statistics. If only the ○ system occurs in nature, this gives us cause on the one hand to reduce the statistical weight just in the sense meant by BOSE; and, on the other hand, by the proper selection of the orbits, PAULI's exclusion of equivalent orbits is coincidently satisfied. The generalization of these results to $n$ component systems proceeds in the following manner. The $n!$ permutations of $n$ systems yield in general an interaction energy with the same eigenvalue. But, by cause of the interaction this degeneracy is lifted; the term chart splits into $n!$ partial systems. Among them, it turns out, there is one

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[8]A. H. COMPTON, *J. Frankl. Inst.* **192**, 145 (1924); E UHLENBECK and S. GOUDSMIT, *Naturwiss.* **13**, 953 (1925).

system that has no equivalent orbits and cannot be combined with all the others, it is this system that occurs in nature and constitutes the real solution. Simultaneously, there is a reduction of the statistical weight from $n!$ to 1 for the BOSE-EINSTEIN enumeration. The formulation described here, therefore, goes beyond the BOSE-EINSTEIN one insofar as it selects from the $n!$ possibilities a particular solution, which involves no equivalent orbits, and therefore satisfies PAULI's exclusion principle. A reason that just this system of all the possible quantum solutions arises, cannot be found in simple quantum calculations. It seems to me, nevertheless, to be an important result of this study, that PAULI's exclusion principle and BOSE-EINSTEIN statistics have the same origin, and that they do not contradict QM. In addition, a paradox emphasized often by EINSTEIN has an analogue in our considerations: when the systems to be coupled are distinct, then classical statistics must be valid for them, in principle up to vanishingly small differentiations. In spite of this, the counting method for differentiated systems is discontinuously different. For differentiated systems the calculations given herein must remain valid, because transformations between the $n!$ systems do occur; thus, no component system can be excluded. These transformations however, occur ever more seldom by cause of diminishing differences between the particles. When the transition amplitudes get smaller than that finite size defined by the sharpness of the state, then, logically it becomes possible to completely exclude transitions and thus to require changing the counting method. It must be emphasized, that these considerations imply that a finite interaction among the component systems is necessary to get a change in the counting. When the periods of the of this resonance effect become longer than the lifetime of the state, obviously, these calculations lose their meaning.

We have emphasized above, that exclusion of all term systems up to a particular one is possible without violating any quantum principle. One must note, however, that this brings with it certain characteristic restrictions. In particular it is that this exclusion implies that it makes no physical sense to talk about the motion of a single electron or of the motion of atomic electrons specified by a matrix of a nonsymmetric function. In general, such a matrix would contain terms that correspond to transitions between subsystems that in fact do not occur in nature. Therefore, for example, for the nonsymmetric case, the exchange relationship [commutation relationship] in its usual form cannot be given a physical interpretation; in contrast to



Fig. 5

a symmetric form of this condition. This is only a formal restriction, however, insofar as in principle all observable intensities result essentially from symmetric functions of the electrons.

The scheme described here has to be taken as programmatic, so long as the mathematical analysis of systems of $n$ identical particles has not been completed. I would like therefore, to add a brief explanation of how term systems not involving equivalent states that can constitute a unique solution can be constructed.

Consider $n$ completely identical subsystems which are in stable states specified by the quantum numbers $m_1, m_2, ..., m_n$. Thus, all total systems consisting of permutations of the $n$ subsystems have the same total energy. The interaction energy, $\mathbf{H}^1$, to first approximation contains only terms that correspond to transitions from at most two subsystems when $\mathbf{H}^1$ is composed of the sum of interactions of two subsystems. To render the matrix of the time mean energies, $\mathbf{W}^1$, diagonal, as in §2, we must find a canonical transformation matrix. The determination of this matrix is actually a well known problem in finding the solution
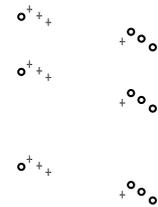
to a system of $n!$ equations with $n!$ unknowns, denoted $S_k$. Let us denote with $H^1_\alpha$ that term in $\mathbf{H}^1$ that corresponds to no transitions between subsystems; and, those terms that correspond to an exchange of two of the quantum numbers $m_1, m_2, ..., m_n$, by $H^1_\beta, H^1_\gamma, ...H^1_\nu$ of which there are $n(n-1)/2$ in number. Then, let us order the $n!$ terms representing equal energies, such that the first position is for the state $m_1, ..., m_n$, then all states, which are obtained through one "transposition"; thereafter those which from the initial distribution, two transposition are needed, etc. The characteristic determinate for the $n!$ linear equations then takes the form:

(18)
$$\begin{vmatrix} W - H^1_\alpha & -H^1_\beta & -H^1_\gamma & ... & -H^1_\nu & 0....0 \\ -H^1_\delta & W - H^1_\alpha & 0 & ... & 0 & -H^1_\varepsilon.... \\ -H^1_\eta & 0 & W - H^2_\alpha & 0 & ..... & -H^1_\xi.... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \end{vmatrix}$$

A solution for this set of equations is:

(19)
$$W^1 = H^1_\alpha - H^1_\beta - H^1_\gamma - ... - H^1_\nu; \quad S_k = \frac{1}{\sqrt{n!}}(-1)^{\delta_k},$$

where $\delta_k$ denotes the number of transpositions that are necessary in order to get from the state $m_1, ..., m_n$ to that state with the number $k$. The corresponding SCHRÖDINGER wave eigenfunction is

(20)
$$\varphi = \frac{1}{\sqrt{n!}}\sum(-1)^{\delta_k}\varphi_1(m^k_\alpha)\varphi_2(m^k_\beta)...\varphi_n(m^k_\nu),$$

where $\delta_k$ again is the number of transpositions needed to turn the sequence $m_1 m_2 ... m_n$ into $m^k_\alpha m^k_\beta ... m^k_\nu$. The functions $\varphi$ have the peculiar characteristic that for an exchange of the quantum numbers for the second subsystem, it changes sign. From this fact, it follows immediately that the term system corresponding to Eq. (20) includes no terms within which equivalent states are included. The eigenfunction $\psi$ for such a state must be invariant under exchange of equivalent states. If $f$ is a function (or an operator) representing radiation, and therefore necessarily symmetric, then the integral

(21)
$$\int f\psi\varphi d\Omega = 0,$$

because for the exchange of equivalent subsystems it must both change sign and not change value.

One can see by deduction (reasoning from $n$ to $n+1$) that terms of the type in Eq. (20) actually constitute a closed system, i.e., that they include no equivalent subsystems. However, so far I have not been able to prove this rigorously. With exception of this technicality, Eq. (20) can be considered a definitive construction of the solution's energy level diagram.

After this rather mathematical addendum, let me return to the issue of physical content of this study. The relationship of BOSE-EINSTEIN statistics of identical particles to QM is determined by the choice of a particular mechanical solution among many possibilities. Such a choice for sufficiently small interaction consists essentially in the phase relationships between the particles of the subsystems. Perhaps with deeper investigation one could expose these phase relationships, and they would be seen as direct analogues to interference of DE BROGLIE waves. Possibly even, deeper analysis would reveal that the basis for the selection of the particular solution exists already in the complications typically arising

in coupling problems. To me, however, it seems from the above considerations to follow, that such difficulties need no solution if the goal is to forecast the emission spectrum of atoms. Atomic spectra are determined satisfactorily by QM alone.

Translated by A. F. KRACKLAUER ©2006