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Comment on: Derivation of the Schrödinger equation from Newtonian mechanics

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Nelson's derivation of Schrödinger's equation with a stochastic argument is shown to be compatible with demonstrations that quantum probabilities evolve deterministically.

I. INTRODUCTION

The following issue of the foundations of quantum mechanics mechanics has come to something of a paradoxical deadlock: On the one hand, the work of Nelson and others [1] encourages the inference that stochastic processes play some sort of ill-understood role in quantum phenomena, while on the other hand, Gilson [2] and Hall and Collins [3] have shown that stochastic mechanics and quantum mechanics coincide only in the trivial case of a deterministic stochastic process. The point of this comment is twofold: first, to exhibit a simple argument which leads to the conclusion of the latter authors, and second, to resolve the apparent paradox by drawing attention to certain elements implicit in the assumptions employed by Nelson in his successful derivation of the Schrödinger's equation, and to show, that in spite of the physical motivation, the formalism of Nelson is compatible with the conclusions of Gilson, Hall and Collins.

II. STOCHASTIC MODELS OF QUANTUM THEORY

The simple argument is in the form of a counter example to the claim that quantum mechanics can be modeled by a Gaussian stochastic process and proceeds as follows: Consider the wave function

$$\psi(x,0) = Ae^{-x^2/2}. \tag{1}$$

Use Feynman's propagator to calculate $\psi(x, t)$:

$$\psi(x,t) = \sqrt{\frac{m}{i2\pi\hbar t}} \times \int dx' e^{[im(x-x')^2/(2\hbar t)]} \psi(x, 0). \tag{2}$$

Now compute $\rho(x,t)$,

$$\rho(x,t) = \psi^*(x, t)\psi(x, t), \tag{3}$$

and compare this with $\rho(x,t)$ computed using a transition probability for a Gaussian stochastic process as follows:

$$\rho(x,t) = \sqrt{\frac{m}{2\pi\hbar t}} \times \int dx' e^{[-m(x-x')^2/(2\hbar t)} \rho(x, 0), \tag{4}$$

where, using Eq. (1, is:

$$\rho(x,0) = \psi^*(x,0)\psi(x,0). \tag{5}$$

Eq. (3) has the general form:

$$\rho(x,t) = \sqrt{\frac{1}{1+at^2}} e^{[-x^2/(1+at^2)}, \tag{6}$$

whereas Eq. (4) results in:

$$\rho(x,t) = \sqrt{\frac{1}{1+at}} e^{[-x^2/(1+at)}. \tag{7}$$

Eq. (6) is the familiar spreading wave packet of quantum theory, whereas Eq. (7) is familiar from diffusion theory.

The conclusions to be drawn from this are, that a Gaussian stochastic process in configuration space (an Einstein-Smoluchowski process [4]) cannot model quantum effects; and, since the very same argument can be executed in momentum space, Ornstein-Uhlenbeck processes (Gaussian processes in velocity variables [4]) are also excluded. Thus, we have counterexamples to the claim that stochastic and quantum mechanics are equivalent. Although this exercise suffers, as does every counterexample, from a paucity of insight into the issue at hand, we offer it only as an expedient alternative to the more revealing but complex arguments of Gilson, Hall and Collins.

III. NELSON'S FORMALISM

Nelson's formulation appears to stand in defiance of the works of Gilson, Hall and Collins. Hall has shown that no stochastic process can model quantum theory, yet Nelson appears to have successfully argued that the Schrödinger equation can be founded on a stochastic process. This conflict can be resolved by showing that the only physical interpretation consistent with the basic assumptions of Nelson's argument, and consistent with classical physics notions, implies that that, in fact, there is no dispersive random process—such as the random impacts responsible for Brownian motion, or by showing an inconsistency.

Nelson's very elegant derivation of the Schrödinger equation is difficult to capture and encapsulate in succinct physical terms. To begin, however, a Markoff process on configuration space is posited:

$$dx = b(x(t),t) + dw(t), \tag{8}$$

where $b(x(t), t)$ is the mean velocity and $w(t)$ is a Wiener process such that $dw(t)$ is independent of $x(s)$ for $s > t$. For such a process there is a Fokker-Planck equation:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(b\rho) + \nu\nabla^2\rho, \tag{9}$$

and a derivation D, where

$$Df = \left(\frac{\partial}{\partial t} + b\cdot\nabla + \nu\nabla^2\right) f, \tag{10}$$

gives the "propagation," as it were, of f in time if ν is the diffusion constant appropriate for the process $w(t)$; i.e., the expectation of $(dw)^2$ is νdt:

$$E\left((dw)^2\right) = \nu dt. \tag{11}$$

This much is clear. However, as Nelson observes, this description is asymmetrical in time, so that "backward" correspondents for all the above expressions may be written:

$$dx = b_*(x(t),t) + dw_*(t); \tag{12}$$

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(b_*\rho) + \nu\nabla^2\rho; \tag{13}$$

$$D_* f = \left(\frac{\partial}{\partial t} + b_*\cdot\nabla\nu\nabla^2\right) f, \tag{14}$$

where, of course, b_* is the mean backward velocity and $dw_*(t)$ is independent of $x(s)$ for $s < t$. This much is also clear in the sense of being mathematically consistent, at least.

Hereafter the argument continues by combining expressions from the forward and backward equations, with the definition of the mean acceleration:

$$a = \frac{1}{2}(D_*D + DD_*)x(t), \tag{15}$$

being crucial.

Little physical insight can be gained by examining mathematical formalism. Physical interpretation is relegated to the beginning and end of a mathematical development. In the case of this 'stochastic' process, Eqs (8) and (12) are at the foundation where physical interpretation can be fruitful. These expressions are kinematical, and, as such, should be amenable to "physical" comprehension if this process is to be used in "an entirely classical derivation and interpretation of the Schrödinger equation." [1]. In this regard, Eq. (8) offers no difficulty; however, the backward version, Eq. (12), especially in combination with forward equation, has been given no physical rationalization.

If an attempt is made to ascribe physical meaning to the backward equation, one of two possible conclusions emerges: Either the process is not a stochastic process, or the world is incomparable more bizarre than can be imagined from classical physics, if not altogether irrational. The most direct interpretation possible is, that Eq. (12) is, in fact, Eq. (8) expressed in a time-reflected frame. No physical process is implied, only the parameter space used in the mathematical model is reflected. The physical process in each case is identical; the Wiener process $w(t)$ which seeks to describe the physical process is also identical, only the parameter t is reflected. All this implies that the backward process from the forward point of view is the inverse of forward process. Clearly then, if the two process are combined in one frame, the total effect is equivalent to no process at all. Although the argument can rendered mathematically, nothing more, including clarity, is thereby won.

Closer examination of Nelson's manipulations reveals that the average forward and backward velocities at a given point in space and time are not equal in magnitude. Therefore, the just given interpretation is inconsistent with Nelson's argument, in additions to leading to a vacuous stochastic process. Nevertheless, if dw_* is regarded as the time reversal of dw, it follows that the combination will result in a process which is not Gaussian. Since the only assumption regarding the backward equation, that does not engender severe difficulties with ontology from classical physics, as is argued below, is that it is the forward equation expressed in a time-reflected frame, the above analysis should be particularly incisive with respect to the source of randomness which the Wiener process is to describe. Therefore, it appears that ethereal Brownian motion, as described by Nelson, is a figure of speech which cannot be accepted as physically coherent.

Having exhausted the direct interpretation, we must look further. Individually, Eqs. (8) and (12) can be given physical meaning in two ways without assuming any relationship between the processes $w(t)$ and $w_*(t)$ (in which case there are two independent processes). Either the dx in each equation is taken to be identical and it is assumed that each equation holds intermittently, or it is assumed that there are in fact two ensembles of sample paths (essentially two particles), one for each process. Each of these interpretations is bizarre and plagued with paradox. For example, it is unclear why the backward process cannot be expressed in a forward frame and added to what is there already to give an ordinary Gaussian process of the sort discussed above in the case where it is assumed that the two equations hold intermittently. In the second case it is unclear how observers (also quantum systems) can interact independently with both ensembles in order to observe the total quantum phenomena resulting from the combination of these two independent ensembles. Other interpretations are also possible; but, they also violate the identity of individual sample paths and, therefore, are in conflict with classical physics.

IV. CONCLUSION

Nelson's process does not admit a physical interpretation which is a natural development of reasoning used in classical physics. His process and derivation of Schrödinger's equation can be regarded only as a formalistic reduction of quantum theory to stochastic theory.

These observations give support to the viewpoint expressed by Gilson, that “quantum mechanics has little if anything to do with stochastic theory,” [2] by exposing inadmissible physical motivation for an otherwise successful formalism.

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