

## RECIPROcity IN THE MUTUAL INTERACTION OF CHARGED PARTICLES

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*A translation of:*  
**Wederkeerigheid in de Werking van Geladen Deeltjes**  
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ABSTRACT. It is shown that a certain reciprocity in the *mutual interaction of two charged particles* can be obtained by assuming that the motion of particle 1 is guided by the *retarded* potentials from particle 2, the motion of 2 being guided by *advanced* potentials from 1. Indeed, if in the integrals of the *variation principle*

$$0 = \delta \left[ \int -m_1 c ds_1 + \int -m_2 c ds_2 + -\frac{e_1 e_2}{4\pi c} \frac{(d\mathbf{x} \cdot d\mathbf{y})}{(\mathbf{R} \cdot d\mathbf{x})} \right],$$

the elements  $d\mathbf{x}$  and  $d\mathbf{y}$  of the first and second particle's tracks respectively and connected by  $(1+3)$ -dimensional radii  $\mathbf{R}$ , of magnitude  $\mathbf{R} \cdot \mathbf{R} = 0$ , so that  $\mathbf{R} \cdot d\mathbf{x} \equiv \mathbf{R} \cdot d\mathbf{y}$ , then the interaction part of this principle does not change by interchanging  $d\mathbf{x}$  and  $d\mathbf{y}$ .

From this variation principle it follows that the system has the *energy and momentum integrals*

$$m_1 c \frac{d\mathbf{x}}{ds_1} + \frac{e_1 e_2}{4\pi c} \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} + m_2 c \frac{d\mathbf{y}}{ds_2} + \frac{e_1 e_2}{4\pi c} \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}} - \frac{e_1 e_2}{4\pi c} \mathbf{R} \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})},$$

the last term not being attributable to either particle in particular. These equations are in no way connected with any consideration of a field.

In classical mechanics, one determines equations of motion using a variation principle. In order to describe the motion of a particle in a force field, one forms from the kinetic energy,  $T$ , and the potential energy,  $U$ , the so-called LAGRANGE function:

$$L = T - U,$$

and then, along the trajectory of the particle between two fixed space-time points, evaluates the integral:

$$\int_1^2 L dt,$$

which is known as HAMILTON's "principle function."

The motion is varied, in that we for each space-time point,  $t, x, y, z$ , [i.e.,  $t, \vec{x}$ ], for a small time displacement  $\delta t$  also displace  $\vec{x}$  to  $\delta \vec{x}$ . The variations,  $\delta t$  and  $\delta \vec{x}$  should be arbitrary functions of a parameter that specifies the location of motion, e.g., the time. If these variations at the beginning and end of the considered interval are made to vanish, then it should be true that, if the motion is natural, the principle function is left unaltered. Executing the variation, in which partial integrations find use, gives

$$\delta \int L dt = \int dt \left\{ \frac{\partial L}{\partial t} - \frac{d}{dt} \left( L - \sum_{\text{cycl.}} \dot{x} \frac{\partial L}{\partial \dot{x}} \right) \right\} \delta t + \sum_{\text{cycl.}} \int dt \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right\} \delta x.$$

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Dots over a symbol denote a differentiation with respect to time.

The demand that this should be null, whichever variation one chooses, leads to the equation of LAGRANGE, i.e.:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0, \text{ CYCL.},$$

and to an equation, that already is a result contained in the preceding:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left( L - \sum_{\text{cycl.}} \dot{x} \frac{\partial L}{\partial \dot{x}} \right) = 0,$$

in which the change of energy ( $-L + \sum \dot{x} \partial L / \partial \dot{x}$ ) is determined in a variable force field.

HAMILTON's principle function is defined so as not to depend on which coordinate system is used to describe the motion. If one has more than one particle, then, in place of Cartesian coordinates, one can use the mutual separation along a line joining the particles.

The coordinates should appear, of course, in the potential energy,  $U$ , which, by definition, can not depend on velocities. However, the kinetic energy in the LANGANGian can depend on the coordinates.

If there are various particles that interact with each other, then the classical theory adds them all together into a general potential energy in which they all take part. There is reciprocity in the interaction between two particles when the potential energy fully depends in the same way from the coordinates of each.

For the mutual interaction by cause of gravitation, the potential energy of two particles is  $\alpha m_1 m_2 / r$ , where  $r$  is their separation. In fact,  $r$  is a fully symmetric function of  $\vec{x}_1$  and  $\vec{x}_2$ , so that  $\partial L / \partial \vec{x}_1 = -\partial L / \partial \vec{x}_2$ . If one, however, must consider in place of  $r$  another function,  $\rho$ , say where:  $\rho^2 = (\beta \vec{x}_1 - \alpha \vec{x}_2)^2$ , then full reciprocity would be destroyed.

In the theory of relativity, in place of the kinetic energy,  $T dt$ , in the integrand of the principle function, one writes:

$$-mc^2 \sqrt{1 - v^2/c^2} dt = -mc \sqrt{c^2 dt^2 - d\vec{x} \cdot d\vec{x}},$$

which, for low values of velocity, goes over to the classical form, up to an additive constant on the factor multiplying  $dt$ .

In place of the potential energy,  $-U dt$ , one must write a differential in the integrand that is a scalar, which, for example, can be achieved with

$$-U dt + \vec{A} \cdot d\vec{x}.$$

This implies the introduction of a *potential momentum* besides the potential energy. Thus, one writes for the modified principle function

$$\int \{ -mc \sqrt{c^2 dt^2 - d\vec{x} \cdot d\vec{x}} - U dt + \sum_{\text{cycl.}} \vec{A} \cdot d\vec{x},$$

again in the form  $\int L dt$ ; so that one then finds for the equations of motion:

$$\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{m \vec{v}}{\sqrt{1 - v^2/c^2}} + \vec{A}.$$

We wish here now to note, that it is not in the spirit of the theory, for the parameter that gives the progress of the motion, to chose 'time.' Let us in stead denote it as  $u$ , and write

$$\frac{dt}{du} = t', \quad \frac{d\vec{x}}{du} = \vec{x}',$$

then we achieve full (3 + 1) dimensional harmony, if we write

$$Ldt = Mdu,$$

and for energy

$$-\frac{\partial M}{\partial t'} = \frac{mc^3 dt}{\sqrt{c^2 dt^2 - d\vec{x} \cdot d\vec{x}}} + U,$$

and for the momentum:

$$\frac{\partial M}{\partial \vec{x}'} = \frac{mc d\vec{x}}{\sqrt{c^2 dt^2 - d\vec{x} \cdot d\vec{x}}} + \vec{A},$$

where  $U$  and  $\vec{A}$  are potential energy and momentum respectively.

If the moving particle carries charge  $e$ , and if the scalar potential in which it moves is  $\phi$  and the vector potential is  $\vec{a}$ , then we know that the potential energy of this particle is

$$U = e\phi,$$

and the momentum is:

$$\vec{A} = \frac{e}{c} \vec{a}.$$

The potentials  $\phi$  and  $\vec{a}$  found in the environment of a charged particle are given by the formulas of LIÉNARD and WIECHERT.

If the trajectory of such a particle is given along  $l$  (see Fig. 1), and we wish to determine the potential at point  $P$ , then first we must identify the space-time emission point  $Q'$ , such that from it a light signal passes through the point  $P$  via a null space-time interval. We can then say that there is 'signal contact' between these two points. The time span,  $\tau$ , and the space interval,  $\vec{\xi}$  between  $P$  and  $Q'$ , i.e.:

$$t_P - t_Q = \tau \quad \vec{x}_P - \vec{x}_Q = \vec{\xi},$$

must then satisfy

$$c^2 \tau^2 - \vec{\xi} \cdot \vec{\xi} = 0.$$

When one has found for the point  $P$ , the appropriate source point  $Q'$ , then for a charge,  $e$ , with velocity  $\vec{v}$ , the [LIÉNARD and WIECHERT] potentials are:

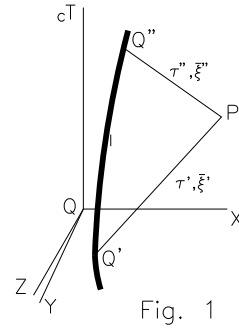
$$\phi = \frac{1}{4\pi} \frac{e}{r(1 - v_r/c)}, \quad \vec{a} = \frac{1}{4\pi c} \frac{e\vec{v}_Q}{r(1 - v_r/c)}.$$

Considering that  $r = c\tau$ , and  $r\vec{v}_r = (\vec{\xi} \cdot d\vec{x})/dt$ , from this we get

$$\phi = \frac{ec}{4\pi} \frac{dt}{c^2 \tau dt - \vec{\xi} \cdot d\vec{x}}, \quad \vec{a} = \frac{e}{4\pi} \frac{d\vec{x}}{c^2 \tau dt - \vec{\xi} \cdot d\vec{x}}.$$

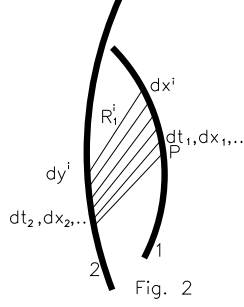
In the denominator there is what one calls a scalar product of (1 + 3)-dimensional vectors  $ds$ , i.e., the differential of motion along the trajectory of the charge, with the contact signal from  $Q$  to  $P$ .

Above we have taken the delayed potentials. However, one might just as well take advanced potentials. To do so, one must find the space-time point  $Q''$  with which  $P$  could



have signal contact, but here originating at  $P$ . Denoting all corresponding quantities for this case with double primes, we get (using  $\tau'' = -r''/c$ )

$$\varphi'' = \frac{ec}{4\pi} \frac{dt''}{c^2\tau''dt'' - \vec{\xi}'' \cdot d\vec{x}''}, \quad \vec{a}'' = \frac{e}{4\pi} \frac{d\vec{x}''}{c^2\tau''dt'' - \vec{\xi}'' \cdot d\vec{x}''}.$$



We wish to use the potentials in order to formulate the principle function for the purpose of describing the motion of a charged particle  $m_1, e_1$  in the ‘field’ of a second particle  $m_2, e_2$ , for which the motion is known.

We focus here on the retarded potentials.

If we divide up the motion of the first particle into differential elements,  $dt_1, d\vec{x}_1$ , then we must find the contact signals from the ends of these elements to those on the trajectory of the second particle. The differential elements with which these elements divide up the second particle’s trajectory (See Fig. 2) we call  $dt_2, d\vec{x}_2$ . The contact rays have the components  $\tau, \vec{\xi}$ .

Now, given the just found potentials substituted into the principle function formulation, we find it can be written as:

$$\int \left\{ -mc ds_1 - e_1 \varphi dt_1 + \frac{e_1}{c} \vec{a} \cdot d\vec{x}_1 \right\} = \int \left\{ -mc ds_1 - \frac{e_1 e_2}{4\pi c} \frac{c^2 dt_2 dt_1 - d\vec{x}_1 \cdot d\vec{x}_2}{c^2 \tau dt_2 - \vec{\xi} \cdot d\vec{x}_2} \right\}.$$

In this integral there are only such terms that for arbitrary transformations of motion, are invariant. Herein  $ds_1$  is described with words in the past tense. The denominator can also be written differently, however. Wherever all the included terms represent signal contact,  $R \equiv 0$ , i.e.:

$$0 \equiv dR^2 = c^2 \tau (dt_1 - dt_2) - \vec{\xi} \cdot (d\vec{x}_1 - d\vec{x}_2),$$

so that the denominator is:

$$c^2 \tau dt_2 - \vec{\xi} \cdot d\vec{x}_2 = c^2 \tau dt_1 - \vec{\xi} \cdot d\vec{x}_1.$$

This means that in the second term of the integral it makes no difference whether the index is 1 or 2, and they can be freely exchanged.

Finally, for a prescribed motion of charge 1, to find the equation of motion for charge 2, we must again determine a similar principle function.

If we are to describe this motion under the retarded influence from charge 1, the non-kinetic part would be specified by this just developed term. Thus, for the mutual interaction of these two identical particles for the principle function take the expression:

$$\int \left\{ -m_1 c ds_1 - m_2 c ds_2 - \frac{e_1 e_2}{4\pi c} \frac{c^2 dt_1 dt_2 - d\vec{x}_1 \cdot d\vec{x}_2}{c^2 \tau dt_2 - \vec{\xi} \cdot d\vec{x}_2} \right\},$$

where, it should be mentioned, that the last term is fully symmetric in the elements of motion for the two particles in so far as an exchange of the indices changes nothing. Here, the contact signal  $\tau, \vec{\xi}$  is taken from 2 to 1.

From these considerations, we find a certain mutuality in the equations of motion. To maintain mutuality demands that, for a delayed interaction departing from 2 and affecting 1, we must consider an advanced interaction from 1 that affects 2.

We have joined thereby each differential of motion mutually with those that are in signal contact. That is also an invariant coordinate for the two motions, which gives a solution to the question: how shall the theory of relativity treat systems of more than one charge.

One could object now also to asymmetry, in that one particle is affected only by the retarded, while the other is affected only by an advanced potential. This can be remedied in the principle function by taking the sum of half the contribution as retarded and half as advanced, thusly:

$$\int \left\{ -m_1 c ds_1 - m_2 c ds_2 - \frac{e_1 e_2}{8\pi c} \frac{c^2 dt_1 dt_2' - d\vec{x}_1 \cdot d\vec{x}_2'}{c^2 \tau dt_2' - \vec{\xi} \cdot d\vec{x}_2'} - \frac{e_1 e_2}{8\pi c} \frac{c^2 dt_1 dt_2'' - d\vec{x}_1 \cdot d\vec{x}_2''}{c^2 \tau dt_2'' - \vec{\xi} \cdot d\vec{x}_2''} \right\}.$$

A glance at Fig. 3 should indicate the meaning of the primes. The variables  $\tau''$ ,  $\vec{\xi}''$  form the contact ray that runs from  $ds_1$  to  $ds_2''$ . This is in contrast to its meaning in one of the above formulae.

It should be clear that the sum of the terms in the integral does not depend, whether one takes the integration parameter to be the interval along one or the other trajectory.

In the following we want, however, an evaluation of the amount of energy and the momentum devolving from the specific choice of a single term in the principle function for the interaction.

Now, we bundle as a single system charge 1 at space-time point  $t_1, \vec{x}_1 \equiv \mathbf{x}$ , and charge 2 at  $t_2, \vec{x}_2 \equiv \mathbf{y}$ , linked together with the contact ray:  $\mathbf{R} \equiv \mathbf{x} - \mathbf{y}$ , where  $\mathbf{R} \cdot \mathbf{R} = 0$ .

As a consequence we shall express the variation as follows:

$$0 = \delta \left[ \int \left\{ -m_1 c ds_1 - m_2 c ds_2 - \frac{e_1 e_2}{4\pi c} \frac{d\mathbf{x} \cdot d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} \right\} \right].$$

Here we used the abbreviated notation  $d\mathbf{x} \cdot d\mathbf{y}$  and  $\mathbf{R} \cdot d\mathbf{y}$  in place of the fully written out scalar products as presented above.

The integral must be taken over two intervals of motion one from each charge as bounded by common contact rays. For the variation, care must be taken that displacements between a pair of differential elements belonging to each other, that the 4-space interval remains null. This happens when from arbitrary variations  $\delta_1 \mathbf{x}$  and  $\delta_2 \mathbf{y}$  one formulates the variations

$$\begin{aligned} \delta \mathbf{x} &= \delta_1 \mathbf{x} + \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}} (\mathbf{R} \cdot \delta_2 \mathbf{y}), \\ \delta \mathbf{y} &= \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} (\mathbf{R} \cdot \delta_1 \mathbf{x}) + \delta_2 \mathbf{y}. \end{aligned}$$

One sees easily from these expressions that:

$$\delta \mathbf{R}^2 = \mathbf{R} \cdot \delta \mathbf{x} - \mathbf{R} \cdot \delta \mathbf{y} \equiv 0.$$

Always and forever one may write  $\mathbf{R} \cdot d\mathbf{y} = \mathbf{R} \cdot d\mathbf{x}$ , and we should make use of this, in so far as when we execute a variation of  $\delta_1 \mathbf{x}$ , the numerator factor  $\mathbf{R} \cdot d\mathbf{x}$  is to be varied, and in so far as  $\delta_2 \mathbf{y}$  goes, we must consider the factor  $\delta(\mathbf{R} \cdot d\mathbf{y})$ .

The kinetic portions of the integral give, after a partial integration:

$$\delta \int (-m_1 c ds_1 - m_2 c ds_2) = \int \left\{ \delta_1 \mathbf{x} \cdot d \left( m_1 c \frac{d\mathbf{x}}{ds_1} \right) + \delta_2 \mathbf{y} \cdot d \left( m_2 c \frac{d\mathbf{y}}{ds_2} \right) \right\}.$$

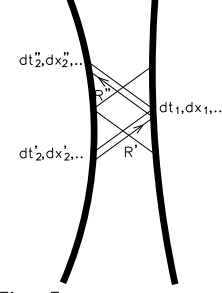


Fig. 3

The interaction term gives:

$$\delta \int \frac{e_1 e_2}{4\pi c} \frac{d\mathbf{x} \cdot d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} = -\frac{e_1 e_2}{4\pi} \int \left[ \frac{d\mathbf{y} \cdot d\delta\mathbf{x} + d\mathbf{x} \cdot d\delta\mathbf{y}}{\mathbf{R} \cdot d\mathbf{x}} - \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} \{ \delta_1(\mathbf{R} \cdot d\mathbf{x}) + \delta_2(\mathbf{R} \cdot d\mathbf{y}) \} \right].$$

For the expansion, it is necessary only to carry out partial integrations, the ultimate result is:

$$\begin{aligned} &= \frac{e_1 e_2}{4\pi c} \int \delta_1 \mathbf{x} \cdot \left( d \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} + \mathbf{R} \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} \cdot d \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}} + \right. \\ &\quad \left. + d\mathbf{x} \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} - \frac{\mathbf{R}}{\mathbf{R} \cdot d\mathbf{x}} \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} - d \frac{\mathbf{R}(d\mathbf{x} \cdot d\mathbf{y})}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} \right) + \\ &\quad + \delta_2 \mathbf{y} \cdot \left( d \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}} + \mathbf{R} \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}} \cdot d \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} - \right. \\ &\quad \left. - d\mathbf{y} \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} + \frac{\mathbf{R}}{\mathbf{R} \cdot d\mathbf{y}} \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} - d \frac{\mathbf{R}(d\mathbf{x} \cdot d\mathbf{y})}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} \right). \end{aligned}$$

This variation is then to be combined with that for the kinetic terms. Finally, as a result of the full variation, the eight complex terms multiplied by  $\delta_1 \mathbf{x}$  and  $\delta_2 \mathbf{y}$  vanish.

Dividing by  $ds_1$ , and  $ds_2$  respectively, gives the equations of motion for both charges.

The expansions indicated by  $d$  have, for both charges, *no* significance as simultaneous elements, rather as two elements with ends linked by contact rays.

We can add up the multiple terms that are multiplied with  $\delta_1 \mathbf{x}$  and, respectively,  $\delta_2 \mathbf{y}$ , and that each by itself is null. In this way four terms that constitute a complete derivative, can be assembled to get:

$$0 = d \left\{ m_1 c \frac{d\mathbf{x}}{ds_1} + \frac{e_1 e_2}{4\pi c} \frac{d\mathbf{y}}{\mathbf{R} \cdot d\mathbf{y}} + m_2 c \frac{d\mathbf{y}}{ds_2} + \frac{e_1 e_2}{4\pi c} \frac{d\mathbf{x}}{\mathbf{R} \cdot d\mathbf{x}} - \frac{e_1 e_2}{4\pi c} \mathbf{R} \frac{d\mathbf{x} \cdot d\mathbf{y}}{(\mathbf{R} \cdot d\mathbf{x})(\mathbf{R} \cdot d\mathbf{y})} \right\}.$$

This expression constitutes four integrals of motion for this system. They represent the total *energy* and *momentum*. The appearance of the last term is especially noteworthy. While one is inclined to interpret the first, respectively second, pair of terms as the kinetic and potential energy (or impulse) of the first, respectively second, charge, we see that the last term alone is also an energy (and impulse) of the whole system, but that it can not be specifically located in either particle in particular.

As is evident from the above, it is possible to develop a dynamics for point charges, that conforms with the requirements of relativity theory, but that does not involve fields. In such point particle dynamics, no energy or momentum is radiated away or otherwise deposited into a field. The two particles themselves are the only actors in the interaction.

We have already remarked, that in order to get full symmetry, we had to take in the above expressions into consideration the contact ray  $\mathbf{R}'$  from 2 to 1 (See: Fig 1), as well as the contact ray  $\mathbf{R}''$  from 1 to 2. If one chooses the parameter  $ds_1$ , then one gets in Figure 3 interactions indicated by double lines as "the system." On the other hand, if one chooses the parameter along the other particle's trajectory, then one takes as the system those interactions indicated with single lines. For either case one can determine an energy and impulse integral, but it is not the same for both cases.

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