

QUANTUM MECAHNICS OF TWO COUPLED SYSTEMS

ERWIN MADELUNG (1881-1972)

A translation from:

Die Mathematischen Hilfsmittel des Physikers

(Dover Publications, New Youk, 1942) pp 298-300..

Section (IV, 4.5.2): Two coupled systems. If V [potential function] has the special form

$$V = V_1(x_1, y_1, z_1) + V_2(x_2, y_2, z_2)$$

then as wave function one may take the form

$$\Psi = \Psi_1(x_1, y_1, z_1)\Psi_2(x_2, y_2, z_2);$$

i.e., the system separates into two fully independent subsystems such that each can be treated as single particle problems.

Frequently, however, there is a small interaction between the subsystems: so that

$$V = V_1(1) + V_2(2) + W(1, 2),$$

where we use the abbreviation $V(1) = V(x_1, \dots)$. The initial solution for a perturbation calculations assumes $W = 0$; i.e.:

$$\begin{aligned} u_i &= u_{(r,s)} = u_r^{(1)}(1)u_s^{(2)}(2), \\ E_i &= E_{(r,s)} = E_r^{(1)} + E_s^{(2)}. \end{aligned}$$

Degeneracy (“*quantum mechanical resonance*”). A degeneracy or quantum mechanical resonance always arises, if for some r, s, r', s' the following holds:

$$E_i = E_{(r,s)} = E_{(r',s')} = E_k.$$

Practical significant is to be found above all by the *special* case of two identical systems

$$V_1 \equiv V_2; \quad \text{so that : } u^{(1)} \equiv u^{(2)} = u,$$

with symmetrical interaction

$$W(1, 2) = W(2, 1).$$

so that $E_i = E_{(r,s)} = E_{(s,r)} = K_k$. The appropriate trial solutions are then:

$$\begin{aligned} u_i &= u_{(r,s)} = u_r(1)u_s(2), \\ u_k &= u_{(s,r)} = u_s(1)u_r(2). \end{aligned}$$

Using the abbreviation

$$\begin{aligned} W_{ii} &= \int W |u_r(1)|^2 |u_s(2)| d\tau_1 d\tau_2, \\ W_{ik} &= \int u_s(1)u_r(2)Wu_r(1)u_s(2) d\tau_1 d\tau_2, \end{aligned}$$

one can write the solution in first approximations as:

$$\begin{aligned} E_i' &= E_i + W_{ii} + W_{ik}, \\ E_k' &= E_i + W_{ii} - W_{ik}, \end{aligned}$$

with the eigenfunctions:

$$u'_i = \frac{1}{\sqrt{2}} (u_r(1)u_s(2) + u_s(1)u_r(2)),$$

$$u'_k = \frac{1}{\sqrt{2}} (u_r(1)u_s(2) - u_s(1)u_r(2)).$$

The first of these is *symmetric*, i.e., invariant under an exchange of particles 1 and 2. The second is *antisymmetric*, i.e., changes sign under this exchange.

Significance of W_{ik} . The energy term splits with the difference $2W_{ik}$. Let us assume that at time $t = 0$ particle 1 is in state r and 2 in state s :

$$\Psi(0) = u_r(1)u_s(2) = \frac{1}{\sqrt{2}}(u'_i + u'_k).$$

Then the equation:

$$\Psi(t) = \frac{1}{\sqrt{2}}(\Psi'_i + \Psi'_k) = \frac{1}{\sqrt{2}} \left(u'_i e^{-\frac{i}{\hbar}E'_i t} + u'_k e^{-\frac{i}{\hbar}E'_k t} \right);$$

holds; rearranging it gives

$$\Psi(t) = e^{\frac{i}{\hbar}(E_i + W_{ii})t} [u_r(1)u_s(2) \cos(W_{ikt}/\hbar) - iu_r(2)u_s(1) \sin(W_{ikt}/\hbar)].$$

Thus, for $W_{ikt}/\hbar = 0, \pi, 2\pi, \dots$ the first term vanishes, while for $W_{ikt}/\hbar = \pi/2, 3\pi/2, \dots$ the second term vanishes and the particles have exchanged places. The frequency of this exchange oscillation is

$$\nu_e = 2W_{ik}/h;$$

where $h\nu_e = 2SW_{ik} = E'_i - E'_k$ is called the exchange energy, and W_{ik} , the exchange integral. Exchange energy has no classical analogue.